Tree-graded Spaces and Relatively Hyperbolic Groups

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Introduction

Geometric group theory studies certain metric graphs, called Cayley graphs, which can be associated to a finitely generated group. To be more precise, the Cayley graph associated to a finitely generated group depends on the choice of a finite system of generators, even though the large scale geometry of the graph does not depend on this choice.

The objects we will be mostly interested in are asymptotic cones of relatively hyperbolic groups. A relatively hyperbolic group is a group with the same "geometric" properties as the fundamental groups of finite volume negatively curved (e.g. hyperbolic) manifolds. There are many interesting examples of this kind of manifolds in dimension 3. For example, many knot complements admit a hyperbolic finite volume structure (these knots are called hyperbolic), as well as many fiber bundles over S^1 with fiber a surface (of negative Euler characteristic). The fundamental group of a finite volume negatively curved manifold is hyperbolic relative to its cusp subgroups. For example, the fundamental group of a hyperbolic knot complement is hyperbolic relative to a subgroup isomorphic to $\mathbb{Z}\oplus\mathbb{Z}$ (corresponding to the boundary of a tubular neighborhood of the knot).

Asymptotic cones are "ways to look at a metric space from infinitely far away". They do not take into any account local properties, but they often provide important information on the large scale geometry. One of the main results which we will prove is the following (not stated here in its full generality).

Theorem 0.1. *If, for $i = 1, 2$, G_i is hyperbolic relative to H_i, and H_1 is quasi-isometric to H_2, then asymptotic cones of G_1 and G_2 with the same scaling factor are bilipschitz homeomorphic.*

In particular, the fundamental groups of hyperbolic knot complements have homeomorphic asymptotic cones.

Outline

In Chapter 1, besides recalling a few definitions and setting some notation, we will review basic concepts of geometric group theory.

Chapter 2 is dedicated to a (quite informal) introduction to nonstandard extensions. Nonstandard methods are powerful tools to formally deal with concepts such as "infinitesimals", "infinite numbers", "points infinitely far away", etc.

In Chapter 3 we will define asymptotic cones. The asymptotic cones of a metric space are obtained "rescaling the metric by an infinitesimal factor", in such a way that "infinitely far away" points become close, while points which are not far enough are identified. The definition we will present, based on nonstandard methods, is not the most used one in the literature. In fact, use of nonstandard methods tends to be avoided and a definition based on ultrafilters is usually given, even though the ultrafilters based definition is just a restatement of the nonstandard definition. The author thinks that the nonstandard definition is far more convenient because, besides providing a lighter formalism, it allows to directly apply basic results about nonstandard extensions, particular cases of which ought to be proved in most arguments if the other definition is used. Also, the nonstandard definition is "philosophically" closer to the idea of looking at a metric space from infinitely far away, while the other one is closer to the idea of Gromov of convergence of rescaled metric spaces, which is more complicated to "visualize".

We will provide examples of how nonstandard methods can be used to prove results about asymptotic cones. For example, we will classify the possible real trees appearing as an asymptotic cone of a group.

In Chapter 4 we will deal with tree-graded spaces, as the asymptotic cones we are mostly interested in have a tree-graded structure. These spaces are generalizations of real trees, which are tree-graded with respect to their points. In fact, for example, while there are no simple loops in real trees, all simple loops in a tree-graded space are confined to lie on certain subsets, called pieces.

In Section 4.1, an alternative definition of tree-graded space, based on projections on pieces, will be given. This definition will be useful later, as it turns out that showing that certain spaces are tree-graded according to this definition is easier.

In Chapter 5 we will define asymptotically tree-graded spaces, which are, roughly, spaces whose asymptotic cones are tree-graded. Relatively hyperbolic groups are groups whose Cayley graph is asymptotically tree-graded (with respect to the lateral classes of certain subgroups). We will also present the analogue for asymptotically tree-graded spaces of the definition of tree-graded space given in Section 4.1. Using this definition, after recalling some properties and results about finite volume negatively curved manifolds, we will prove that the fundamental groups of manifolds of this kind are relatively hyperbolic.

In Chapter 6 we will prove Theorem 0.1. We will start by showing some results about the interaction between certain algebraic properties of a relatively hyperbolic group G and the geometry of its asymptotic cone (some purely algebraic results will also be found during the discussion). Using this interaction we will be able to determine the isometry type of certain important subsets of the asymptotic cones of G called transversal trees. A careful study of the geodesics in a tree-graded space follows. We will then "count" how many geodesics of each possible kind start from a given point in an asymptotic cone X of G, and we will show that this is enough to determine the homeomorphism type of X.

In Chapter 7 we will partially answers 2 natural questions which arise from the previous chapter, that is if there always exists a tree-graded space such that the set of geodesics of a certain kind has an assigned cardinality, and if this determines the isometry type of the tree-graded space. We will focus on the case that the assigned cardinalities are infinite, showing that in this case the answer to both questions is affirmative.

Contents

Chapter 1

Basic notation and definitions

If X is a metric space, $x \in X$, $A \subseteq X$ and $r \in \mathbb{R}^+$, we will use the following notation:

- $B_r(x)$ (resp. $\overline{B}(x,r)$) is the open (resp. closed) ball with center x and radius r,
- if explicit mention of X is needed, the same ball will be denoted by $B_X(x,r)$,
- $N_r^X(A) = N_r(A) = \{x \in X : d(x,A) \leq r\}$, is the $r-$neighborhood of A. We will use the first notation only if explicit mention of X is needed,
- A is $r-$dense in X if $N_r(A) = X$.

A metric space X is proper if closed balls in X are compact.

If X is a metric space and $A, B \subseteq X$, the Hausdorff distance between A, B is defined as

$$\inf\{K : A \subseteq N_K(B), B \subseteq N_K(A)\}.$$

It is indeed a distance on the set of compact subsets of X.

Definition 1.1. The metric space X is homogeneous if for each $x, y \in X$ there exists an isometry $f : X \to X$ such that $f(x) = y$. Equivalently, X is homogeneous if there is only one orbit for the action on X of its isometries.

The metric space X is quasi-homogeneous if one (and therefore each) orbit of the action of the isometries of X is $k-$dense, for some $k \geq 0$.

Injective paths will be called arcs. The length of a curve γ will be denoted by $l(\gamma)$. A geodesic (parametrized by arc length) in the metric space X is a curve $\gamma : [0,l] \to X$ such that $d(\gamma(t),\gamma(s)) = |t-s|$ for each $t,s \in [0,l]$. The most frequently used property of geodesics is that if γ is a geodesic for $t \leq s$ we have $l(\gamma|_{[t,s]}) = d(\gamma(t),\gamma(s))$. The metric space X is geodesic if for each $x,y \in X$ there exists a geodesic from x to y. With an abuse, geodesics will frequently be identified with their images and if $x,y \in X$ and X is geodesic we will sometimes denote by $[x,y]$ a geodesic between them, even though this geodesic need not be unique. A subset A of a geodesic metric space X is called geodesic if for each pair of points in A there is a geodesic connecting them contained in A, and it is called convex if each geodesic in X connecting 2 points in A is contained in A.

Geodesic rays and lines are defined similarly, that is, the curve $\gamma : I \to X$ such that $d(\gamma(t),\gamma(s)) = |t-s|$ for each $t,s \in I$ will be called geodesic ray if $I = [0,+\infty)$ or $I = (-\infty,0]$ and geodesic line if $I = \mathbb{R}$.

When dealing with asymptotic cones, sometimes we will refer to an object which can be either a geodesic, a geodesic ray or a geodesic line simply as geodesic.

A geodesic triangle is a union of geodesics γ_i, $i = 0,1,2$, called *sides*, such that the final point of γ_i is the starting point of γ_{i+1}. The definition of quasi-geodesic triangle is similar (see the next section for the definition of quasi-geodesics).

Definition 1.2. A geodesic triangle is $\delta-$thin if, denoting by γ_i its sides, $\gamma_i \subseteq N_\delta(\gamma_{i-1} \cup \gamma_{i+1})$.

A geodesic metric space X is called (Gromov-)hyperbolic if there exists δ such that each geodesic triangle in X is $\delta-$thin.

A general reference for hyperbolic spaces is [GdH].

Notice that the definition of $\delta-$thinness can be given also for quasi-geodesic triangles.

Definition 1.3. A tripod is a $0-$thin geodesic triangle.

A real tree is a geodesic metric space such that all its geodesic triangles are tripods.

If T is a real tree and $p \in T$, the valency of T at p is the number of connected components of $T \backslash \{p\}$.

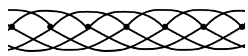
Finally, as it will provide convenient notation, let us review basic aspects of the theory of ordinals and cardinals (see any textbook in set theory, for example [HJ]). Ordinals are certain totally ordered sets (indeed, well ordered sets), and among them there are representatives for each cardinality, called cardinals. There is a total order on ordinals (which coincides with the inclusion), and each ordinal α coincides with the set of ordinals β such that $\beta < \alpha$. Each family $\{\alpha_i\}_{i \in I}$ of ordinals (in particular, of cardinals) has a supremum, denoted $\sup \alpha_i$. If each α_i is a cardinal, $\sup \alpha_i$ is a cardinal. From now on, we will not distinguish a cardinal from the cardinality it represents.

1.1 Geometric group theory

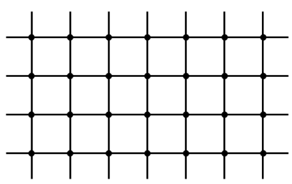
The reader is referred to [Bo] for the proofs of all the results in this section except Theorem 1.7, which is proved in [BH].

Throughout the section G will denote a finitely generated group and $S = S^{-1}$ a finite generating set for G (even if not explicitly stated, we will always assume that generating sets satisfy $S = S^{-1}$). We are going to define a metric graph $\mathcal{CG}_S(G)$ associated to (G, S), which is called the Cayley graph of G with respect to S. The vertices of $\mathcal{CG}_S(G)$ are the elements of G, and there is an edge of length 1 between g and h if and only if there exists $s \in S$ such that $gs = h$. We will consider $\mathcal{CG}_S(G)$ endowed with the path metric induced by this data. We will often identify G with the set of vertices of $\mathcal{CG}_S(G)$. Using this identification, the distance between two elements $g, h \in G$ is the length of the shortest word in the alphabet S which represents $g^{-1}h$. It is easy to prove that $\mathcal{CG}_S(G)$ is a geodesic metric space.

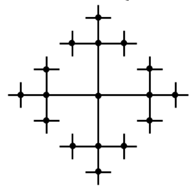
$G = \mathbb{Z}, S = \{\pm 1\}$



$G = \mathbb{Z}, S = \{\pm 2, \pm 3\}$



$G = \mathbb{Z}^2, S = \{\pm(1,0), \pm(0,1)\}$



$G = \mathbb{Z} * \mathbb{Z} = \langle a, b \rangle, S = \{a^{\pm 1}, b^{\pm 1}\}$

Changing generating set can change drastically the "local" structure of the Cayley graph, but in some sense the large scale structure remains the same. To make this more precise we need the notion of quasi-isometry.

Definition 1.4. Let X, Y be metric spaces, $k \geq 1$, $c \geq 0$. A $(k, c)-$quasi-isometric embedding $f : X \to Y$ is a function such that for each $x_1, x_2 \in X$

$$\frac{d(x_1, x_2)}{k} - c \leq d(f(x_1), f(x_2)) \leq kd(x_1, x_2) + c.$$

If f also satisfies the property that $N_c(f(X)) = Y$, then f will be called a $(k, c)-$quasi-isometry.

A quasi-isometric embedding (resp. quasi-isometry) is a map which is a $(k, c)-$quasi isometric embedding (resp. $(k, c)-$quasi-isometry) for some constants k, c. A quasi-geodesic is a quasi-isometric embedding of an interval $[0, l]$ in a metric space. Quasi-geodesic rays and lines are defined similarly. With the same abuse as in the case of geodesics, quasi-geodesics (rays, lines) will often be identified with their images.

Lemma 1.5. *A quasi-isometric embedding $f : X \to Y$ is a quasi-isometry if and only if there exists $C \geq 0$ and a quasi-isometric embedding $g : Y \to X$ such that $g \circ f$ and $f \circ g$ are at finite distance from the identity maps of X and Y, respectively.*

Using the characterization of quasi-isometries provided by this lemma, it is easily shown that being quasi-isometric is an equivalence relation between metric spaces. This is the "right" equivalence relation in our case, as shown below.

Proposition 1.6. *If S' is another finite generating set for G, then $\mathcal{CG}_{S'}(G)$ and $\mathcal{CG}_S(G)$ are quasi-isometric.*

One may wonder why we defined the Cayley graph instead of just putting the corresponding metric on G, which would be defined up to bilipschitz equivalence instead of up to quasi-isometry. One reason is that Cayley graphs are geodesic metric spaces, and there are many results for this kind of spaces which are therefore readily usable in our context. The second is that quasi-isometries would play a very important role in the theory anyway, thanks to the following theorem.

Theorem 1.7. (Milnor-Svarc Lemma) *Let X be a geodesic metric space and Γ a group acting on X properly and cocompactly (see [BH] for the definitions) by isometries. Then Γ is finitely generated and the map $x \mapsto \gamma(x)$ is a quasi-isometry, for each $x \in X$.*

This theorem implies, in particular, that the fundamental group of a compact Riemannian manifold M is quasi-isometric to the (metric) universal cover of M. This, in turn, implies that compact Riemannian manifolds with isomorphic fundamental group have quasi-isometric universal covers.

A useful feature of $\mathcal{CG}_S(G)$ is that G acts on it.

Proposition 1.8. *The action of G on itself by left multiplication (that is, $g \in G$ acts by $h \mapsto gh$), induces an action by* isometries *of G on $\mathcal{CG}_S(G)$.*

Notice that G is $1-$dense in $\mathcal{CG}_S(G)$ and (the isometry induced by) left multiplication by g of course maps e to g. This immediately implies:

Corollary 1.9. *$\mathcal{CG}_S(G)$ is quasi-homogeneous.*

Another crucial result is the following.

Lemma 1.10. *Group isomorphisms induce quasi-isometries between the corresponding Cayley graphs.*

It is sometimes useful to consider an isomorphism as a quasi-isometry in order to deduce coarse information about it, and then refine this information. This procedure has been applied, for example, in [DS] to determine properties of automorphisms of relatively hyperbolic groups and in [FLS] to prove rigidity results for certain manifolds.

Chapter 2

Nonstandard extensions

For the following chapters we will need basic results about the theory of nonstandard extensions. The treatment will be rather informal, for a more formal one see for example [Go]. Let us start with a motivating example. It is quite evident that being allowed to use non-zero infinitesimals (i.e. numbers x different from 0 such that $|x| < 1/n$ for each $n \in \mathbb{N}^+$) would be very helpful in analysis. Unfortunately, $\mathbb{R}$ does not contain infinitesimals. The idea is therefore to find an extension of $\mathbb{R}$, denoted by ${}^*\mathbb{R}$, which contains infinitesimals. Let us construct such an extension.

Definition 2.1. Let I be any infinite set. A filter $\mathcal{U} \subseteq \mathcal{P}(I)$ on I is a collection of subsets of I such that for each $A, B \subseteq I$

1. If A is finite, $A \notin \mathcal{U}$ (in particular $\emptyset \notin \mathcal{U}$),
2. $A, B \in \mathcal{U} \Rightarrow A \cap B \in \mathcal{U}$,
3. $A \in \mathcal{U}, B \supseteq A \Rightarrow B \in \mathcal{U}$.

   An ultrafilter is a filter satisfying the further property:

4. $A \notin \mathcal{U} \Rightarrow A^c \in \mathcal{U}$.

This is not the standard definition of ultrafilter: the usual one requires only that $\emptyset \notin \mathcal{U}$ instead of property (1), and the ultrafilters not containing finite sets are usually called non-principal ultrafilters. However, we will only need non-principal ultrafilters.

Fix any infinite set I. An example of filter on I is the collection of complements of finite sets. An easy application of Zorn's Lemma shows that there actually exists an ultrafilter $\mathcal{U}$, which extends the mentioned filter. Fix such an ultrafilter. We are ready to define ${}^*\mathbb{R}$.

Definition 2.2. Define the following equivalence relation $\sim$ on $\mathbb{R}^I = \{f : I \to \mathbb{R}\}$:

$$f \sim g \iff \{i \in I : f(i) = g(i)\} \in \mathcal{U}.$$

Let ${}^*\mathbb{R}$ be the quotient set of $\mathbb{R}^I$ modulo this relation.

It is easily seen using the properties of an ultrafilter (in fact, of a filter) that $\sim$ is indeed an equivalence relation. We can define the sum and the product on ${}^*\mathbb{R}$ componentwise, as this is easily seen to be well defined. Using also property (4), we obtain that ${}^*\mathbb{R}$, equipped with this operations, is a field. We can also define an order ${}^*\leq$ on ${}^*\mathbb{R}$ in the following way:

$$[f]\ {}^*\leq [g] \iff \{i \in I : f(i) \leq g(i)\} \in \mathcal{U}.$$

Using the properties of ultrafilters it is easily seen that this is a total order on ${}^*\mathbb{R}$ (property (4) is required only to show that it is total), and that ${}^*\mathbb{R}$ is an ordered field. An embedding of ordered fields $\mathbb{R} \hookrightarrow {}^*\mathbb{R}$ can be defined simply by $r \mapsto f_r$, where f_r is the function with constant value r. We can identify $\mathbb{R}$ with its image in ${}^*\mathbb{R}$. Finally, ${}^*\mathbb{R}$ actually contains infinitesimals. In fact, considering $I = \mathbb{N}^+$ for simplicity, $[n \mapsto 1/n]$ is easily checked to be a non-zero infinitesimal. Also, $[id_{\mathbb{N}}]$ is a positive infinite, that is, it is greater than n for each $n \in \mathbb{N}$.

Notice that in the definition we gave of ${}^*\mathbb{R}$ we can substitute $\mathbb{R}$ with any set X. Doing so, we obtain the definition of ${}^*X$, which can be considered as an extension of X, just as we considered ${}^*\mathbb{R}$ as an extension of $\mathbb{R}$. In the case of ${}^*\mathbb{R}$, we showed that this extension preserves the basic properties of $\mathbb{R}$, i.e. being an ordered field. The idea is that this is true in general, as we will see.

Before proceeding, notice that if $f : X \to Y$ is any function, we can define componentwise a function ${}^*f : {}^*X \to {}^*Y$ (which is well defined), called the nonstandard extension of f. This function coincides with f on (the subset of ${}^*X$ identified with) X. Also relations have nonstandard extensions (see the definition of ${}^*\leq$). Let us give another definition (in a quite informal way), and then we will see which properties are preserved by nonstandard extensions.

Definition 2.3. A formula ϕ is bounded if all quantifiers appear in expressions like $\forall x \in X$, $\exists x \in X$ (bounded quantifiers).

The nonstardard interpretation of ϕ, denoted ${}^*\phi$, is obtained by adding * before any set, relation or function (not before quantified variables).

An example will make these concepts clear: consider

$$\forall X \subseteq \mathbb{N}, X \neq \emptyset\ \exists x \in X\ \forall y \in X\ x \leq y,$$

which expresses the fact that any non-empty subset of $\mathbb{N}$ has a minimum. This formula is not bounded, because it contains "$\forall X \subseteq \mathbb{N}$". However, it can be turned into a bounded formula by substituting "$\forall X \subseteq \mathbb{N}$" with "$\forall X \in \mathcal{P}(\mathbb{N})$". The nonstandard interpretation of the modified formula reads

$$\forall X \in {}^*\mathcal{P}(\mathbb{N}), X \neq {}^*\emptyset\ \exists x \in X\ \forall y \in X\ x\,{}^*\leq y. \qquad (1)$$

These definitions are fundamental for the theory of nonstandard extensions in view of the following theorem, which will be referred to as the transfer principle.

Theorem 2.4. *(Łoš Theorem) Let ϕ be a bounded formula. Then $\phi \iff {}^*\phi$.*

This theorem roughly tells us that the nonstandard extensions have the same properties, up to paying attention to state these properties correctly (for example, replacing "$\forall X \subseteq \mathbb{N}$" with "$\forall X \in \mathcal{P}(\mathbb{N})$"). Easy consequences of this theorem are, for example, that the nonstandard extension $({}^*G, {}^*\cdot)$ of a group $(G, \cdot)$ is a group, or that the nonstandard extension $({}^*X, {}^*d)$ of a metric space (X, d) is a ${}^*\mathbb{R}-$metric space (that is ${}^*d : {}^*X \times {}^*X \to {}^*\mathbb{R}$ satisfies the axioms of distance, which make sense as ${}^*\mathbb{R}$ is in particular an ordered abelian group). To avoid too many *'s, we will often drop them before functions or relations, for example we will denote the "distance" on ${}^*X$ as above simply by "d", the order on ${}^*\mathbb{R}$ by " $\leq$ " and the group operation on ${}^*G$ by " $\cdot$ ". In view of the transfer principle, the following definition will be very useful:

Definition 2.5. $A \subseteq {}^*X$ will be called *internal* subset of X if $A \in {}^*\mathcal{P}(X)$. An internal set is an internal subset of some ${}^*X$.

$f : {}^*X \to {}^*Y$ will be called *internal* function if $f \in {}^*Fun(X, Y) = {}^*\{f : X \to Y\}$.

One may think that "living inside the nonstandard world" one only sees internal sets and functions, and therefore, by the transfer principle, one cannot distinguish the standard world from the nonstandard world.

Notice that ${}^*\mathcal{P}(X) \subseteq \mathcal{P}({}^*X)$ by the transfer principle applied to the formula

$$\forall A \in \mathcal{P}(X)\ \forall a \in A\ a \in X.$$

Similarly, ${}^*Fun(X, Y) \subseteq Fun({}^*X, {}^*Y)$. Also, $\{{}^*A : A \in \mathcal{P}(X)\} \subseteq {}^*\mathcal{P}(X)$, by the transfer principle applied to $(\forall a \in A,\ a \in X) \Rightarrow A \in \mathcal{P}(X)$, and similarly $\{{}^*f : f \in Fun(X, Y)\} \subseteq {}^*Fun(X, Y)$. To sum up

$$\{{}^*A : A \in \mathcal{P}(X)\} \subseteq {}^*\mathcal{P}(X) \subseteq \mathcal{P}({}^*X),$$

$$\{{}^*f : f \in Fun(X, Y)\} \subseteq {}^*Fun(X, Y) \subseteq Fun({}^*X, {}^*Y).$$

However, the equalities are in general very far from being true, as we will see.

Another example: the transfer principle applied to formula (1), which tells that each non-empty subset of $\mathbb{N}$ has a minimum, gives that each *internal* non-empty subset of ${}^*X$ has a minimum (${}^*\emptyset = \emptyset$ as, for each set A, $\exists a \in A \iff \exists a \in {}^*A$).

Let us now introduce a convention we will often use. For each definition in the "standard world" there exists a nonstandard counterpart. For example, the definition of geodesic (in the metric space X), yields the definition of *geodesics, in the following way. The definition of geodesic (with domain the interval $[0,1]$, for simplicity) can be given as

$$\gamma \in Fun([0,1],X) \text{ is a geodesic } \iff \forall x,y \in [0,1] d(\gamma(x),\gamma(y)) = |x-y|,$$

Therefore the definition of *geodesic can be given as

$$\gamma \in {}^*Fun([0,1],X) \text{ is a } {}^*\text{geodesic } \iff \forall x,y \in {}^*[0,1] d(\gamma(x),\gamma(y)) = |x-y|,$$

Łoš Theorem alone is not enough to prove anything new. In fact, it holds for the trivial extension, that is, if we set $^*X = X$, $^*f = f$ and $^*R = R$ for each set X, function f and relation R. However, the nonstandard extensions we defined enjoy another property, which will be referred to as $\aleph_0-$saturation, or simply saturation. First, a definition, and then the statement.

Definition 2.6. A collection of sets $\{A_j\}_{j\in J}$ has the finite intersection property (FIP) if for each $n \in \mathbb{N}$ and $j_0,\dots,j_n \in J$, we have $A_{j_0} \cap \dots \cap A_{j_n} \neq \emptyset$.

Theorem 2.7. *Suppose that the collection of internal sets $\{A_n\}_{n\in\mathbb{N}}$ has the FIP. Then $\bigcap_{n\in\mathbb{N}} A_n \neq \emptyset$.*

The $\aleph_0$ prefix indicates that the collection of internal sets in the statement is countable. Therefore, for each cardinal k one can define the $k-$saturation similarly. For each k, choosing I and $\mathcal{U}$ carefully, one can obtain nonstandard extensions which satisfy $k-$saturation. This is useful sometimes, but $\aleph_0-$saturation will be enough for our purposes.

Let us use this theorem to prove that $^*\mathbb{R}$ contains infinitesimals. It is enough to consider the collection of sets $\{{}^*(0,1/n)\}_{n\in\mathbb{N}^+}$ and apply the theorem to it. Notice that for $n \in \mathbb{N}^+$, $^*(0,1/n) \in {}^*\mathcal{P}(\mathbb{R})$ as it is of the form $^*A$ for $A \in \mathcal{P}(\mathbb{R})$. More in general, however, for each $x,y \in {}^*\mathbb{R}$, $(x,y) \in {}^*\mathbb{R}$ (we should use a different notation for intervals in $\mathbb{R}$ and intervals in $^*\mathbb{R}$, but hopefully it will be clear from the context which kind of interval is under consideration). In fact, we can apply the transfer principle to the formula $\forall x,y \in \mathbb{R}\ (x,y) \in \mathcal{P}(\mathbb{R})$. To be more formal, "$(x,y) \in \mathcal{P}(\mathbb{R})$" should be substituted by

$$\exists A \in \mathcal{P}(\mathbb{R})\ \forall z \in \mathbb{R}\ (z \in A \iff x < z \text{ and } z < y).$$

We just described a standard strategy to prove that a set is internal, that is, loosely, to prove that it belongs to a family of sets (in our case sets of the form (x,y)) which is defined by the nonstandard interpretation of a formula.

Notice that it can be proved similarly that $^*\mathbb{N}$ and $^*\mathbb{R}$ contain infinite numbers. We will need the following refinement of this:

Lemma 2.8.

1. *Let $\{\xi_n\}_{n\in\mathbb{N}}$ be a sequence of infinitesimals. There exists an infinitesimal ξ greater than any ξ_n.*

2. *Let $\{\rho_n\}_{n\in\mathbb{N}}$ be a sequence of positive infinite numbers (in ${}^*\mathbb{R}$ or ${}^*\mathbb{N}$). There exists an infinite number ρ smaller than any ρ_n.*

Proof. Let us prove (1), the proof of (2) being very similar.

The collection $\{(\xi_n, 1/(n+1))\}_{n\in\mathbb{N}}$ of internal subsets of ${}^*\mathbb{R}$ has the FIP. An element $\xi \in \bigcap(\xi_n, 1/(n+1))$ has the required properties. $\square$

The reader is suggested to forget the definition of nonstandard extensions, as Theorem 2.4, Theorem 2.7 and the remark below are all we need, and the definition will never be used again.

Remark 2.9. The nonstandard extension of a set of cardinality at most $2^{\aleph_0}$ has cardinality at most $2^{\aleph_0}$.

Let us now point out some useful consequences of the transfer principle and saturation.

Proposition 2.10. *If $A \subseteq {}^*X$ is finite, then it is an internal subset.*

The proof is just by induction. Notice that the property of being finite cannot be expressed entirely in the "nonstandard world", in fact:

Remark 2.11. $\mathbb{N} \subseteq {}^*\mathbb{N}$ is not an internal subset.

To prove this, notice that each bounded subset of $\mathbb{N}$ has a maximum, and therefore each bounded internal subset of ${}^*\mathbb{N}$ has a maximum as well. But $\mathbb{N}$ is bounded by any infinite number, while it has no maximum.

Something more general holds:

Proposition 2.12. *Suppose that $A \subseteq X \subseteq {}^*X$ is an internal subset. Then it is finite.*

Now, some lemmas which are frequently used when working with nonstandard extensions. The first one is usually referred to as overspill:

Lemma 2.13. *Suppose that the internal subset $A \subseteq {}^*\mathbb{R}^+$ (or $A \subseteq {}^*\mathbb{N}$) contains, for each $n \in \mathbb{N}$, an element greater than n. Then A contains an infinite number.*

Lemma 2.14. *Suppose that the internal subset $A \subseteq {}^*\mathbb{R}^+$ is such that, for each $n \in \mathbb{N}^+$, $A \cap \{x \in {}^*\mathbb{R} : x < 1/n\} \neq \emptyset$. Then A contains an infinitesimal.*

Lemma 2.15. *Suppose that the internal subset $A \subseteq {}^*\mathbb{R}^+$ is such that, for each positive infinite number ν, $A \cap \{x \in {}^*\mathbb{R} : x < \nu\} \neq \emptyset$. Then A contains a finite number.*

The proofs of these lemmas are very easy. Let us prove the first one, for example.

Proof. The collection of internal sets $\{A\} \cup \{(n, +\infty)\}_{n\in\mathbb{N}}$ has the FIP, therefore $\bigcap_{n\in\mathbb{N}}(n, \infty) \cap A \neq \emptyset$. (For clarity, by (n, ∞) we mean $\{x \in {}^*\mathbb{R} : x > n\}$.) An element in the intersection is what we were looking for.

$\square$

Let us introduce some (quite intuitive) notation, which we are going to use from now on.

Definition 2.16. Consider $\xi, \eta \in {}^*\mathbb{R}$, with $\eta \neq 0$. We will write:

- $\xi \in o(\eta)$ (or $\xi \ll \eta$ if ξ, η are nonnegative) if ξ/η is infinitesimal,
- $\xi \in O(\eta)$ if ξ/η is finite,
- $\xi \gg \eta$ if ξ, η are nonnegative and ξ/η is infinite,
- $\xi \equiv \eta$ if $\xi \in O(\eta) \backslash o(\eta)$.

For example, $o(1)$ is the set of infinitesimals, and $O(1) = \{\xi \in {}^*\mathbb{R} : |\xi| < r \text{ for some } r \in \mathbb{R}^+\}$.

The map we given by the following lemma plays a fundamental role in nonstandard analysis, and will be used in the definition of asymptotic cone:

Proposition 2.17. *There exists a map $st : O(1) \to \mathbb{R}$ such that, for each $\xi \in {}^*\mathbb{R}$, $\xi - st(\xi)$ is infinitesimal.*

We will call $st(\xi)$ the standard part of ξ. Notice that $st(\xi) = 0 \iff \xi$ is infinitesimal.

Many common definitions have interesting nonstandard counterparts. Here are a few examples (some of which will be used later).

Proposition 2.18. *$f : \mathbb{R} \to \mathbb{R}$ is continuous at x if and only if, for each infinitesimal ξ, $({}^*f(x + \xi) - {}^*f(x)) \in o(1)$.*

Proposition 2.19. *Consider $f : \mathbb{R} \to \mathbb{R}$ or $f : \mathbb{N} \to \mathbb{R}$.*

$$\lim_{x\to+\infty} f(x) = l \iff \forall \mu \gg 1 \; |f(\mu) - l| \ll 1.$$

Proposition 2.20. *The sequence of positive real numbers $\{a_n\}$ diverges if and only if for each $\mu \in {}^*\mathbb{N}$, $\mu \gg 1$, $a_\mu \gg 1$.*

Proposition 2.21. *The metric space X is compact if and only if for each $\xi \in {}^*X$ there exists $x \in X$ such that $d(x, \xi) \in o(1)$.*

Notice that the existence of the standard part map follows from this second proposition. Finally, a lemma which is often useful, and which will be used later.

Lemma 2.22. *For any X, any sequence $\{a_n\}_{n\in\mathbb{N}} \subseteq {}^*X$ can be extended to an internal *sequence $\{a_\nu\}_{\nu\in {}^*\mathbb{N}} \subseteq {}^*X$.*

Proof. Let A_n be the set of internal sequences $\{x_\nu\}_{\nu\in {}^*\mathbb{N}}$ such that $x_k = a_k$ for each $k \leq n$. It is readily checked that A_n is internal (we are imposing a "finite condition"). Also, it is non-empty, as the sequence

$$x_\nu = \begin{cases} a_\nu & \text{if} \quad \nu \leq n \\ a_n & \text{if} \quad \nu > n \end{cases}$$

is internal. By saturation, $\bigcap A_n \neq \emptyset$. A sequence in this intersection has the required property.

□

Chapter 3

Asymptotic cones

Let (X, d) be a metric space. The asymptotic cones of X are "ways to look at X from infinitely far away". Let us make this idea precise.

Definition 3.1. Consider $\nu \in {}^*\mathbb{R}$, $\nu \gg 1$. Define on ${}^*X$ the equivalence relation $x \sim y \iff d(x, y) \in o(\nu)$. The asymptotic cone $C(X, p, \nu)$ of X with basepoint $p \in {}^*X$ and scaling factor ν is defined as

$$\{[x] \in {}^*X/\sim: d(x, p) \in O(\nu)\}.$$

The distance on $C(X, p, \nu)$ is defined as $d([x], [y]) = st\left({}^*d(x, y)\right)$.

This definition of asymptotic cone is basically due to van den Dries and Wilkie, see [vDW]. However, the original concept is due to Gromov, see [Gr]. The aim of [vDW] was to simplify the proofs in [Gr].

A trivial example of asymptotic cone is that each asymptotic cone of a bounded metric space is a point (such a space has no large scale geometry). A less trivial one is the following:

Remark 3.2. For each $n \in \mathbb{N}^+$, $p \in {}^*\mathbb{R}^n$ and $\nu \gg 1$, $C(\mathbb{R}^n, p, \nu)$ is isometric to $\mathbb{R}^n$.

Proof. An isometry $C(\mathbb{R}^n, p, \nu) \to \mathbb{R}^n$ is given by

$$[(x_1, \dots, x_n)] \mapsto \left(st\left((x_1 - p)/\nu\right), \dots, st\left((x_n - p)/\nu\right)\right).$$

$\square$

Before proceeding, a few definitions. If $q \in {}^*X$ and $d(p, q) \in O(\nu)$, so that $[q] \in C(X, p, \nu)$, then $[q]$ will be called the projection of q on $C(X, p, \nu)$. Similarly, if $A \subseteq \{x \in {}^*X : d(x, p) \in O(\nu)\}$, the projection of A on $C(X, p, \nu)$ is $\{[a] | a \in A\}$. If $A \subseteq {}^*X$ is not necessarily contained in $\{x \in {}^*X : d(x, p) \in O(\nu)\}$, we will call $\{[a] \in C(X, p, \nu) | a \in A\}$ the set induced by A.

Here are 2 useful properties of asymptotic cones:

Lemma 3.3. *1. Any asymptotic cone is a complete metric space.*

2. If $f : {}^*X \to {}^*Y$ is a ${}^*(k,c)-$quasi-isometric embedding, for some $k, c \in \mathbb{R}^+$, and $d(f(p), q) \in O(\nu)$, then f induces a $k-$bilipschitz map $C(X,p,\nu) \to C(Y,q,\nu)$. If f is a ${}^*(k,c)-$quasi-isometry, the induced map is a $k-$bilipschitz homeomorphism.

3. Any asymptotic cone of a geodesic metric space is a geodesic metric space.

4. If X is quasi-homogeneous, then $C(X,p,\nu)$ is homogeneous for each $p \in {}^*Y$, $\nu \gg 1$.

Proof. (1) Consider a Cauchy sequence $\{[x_n]\} \subseteq C(X,p,\nu)$. By Lemma 2.22, there exists an internal sequence $\{x_\nu\}_{\nu \in {}^*\mathbb{N}}$ which extends $\{x_n\}_{n\in\mathbb{N}}$. We want to show that $[x_\rho]$ is the limit of $\{[x_n]\}$, for some infinite ρ. Indeed, by the fact that $\{[x_n]\} \subseteq C(X,p,\nu)\}$ is Cauchy, we easily get that

$$\forall r \in \mathbb{R}^+ \ \exists m(r) \in \mathbb{N} \ \forall m(r) \leq m_1, m_2 \in \mathbb{N} \ d(x_{m_1}, x_{m_2}) \leq r\nu.$$

By overspill and using the formula above for $r = 1/k$, we have that for each $k \in \mathbb{N}^+$ there exists some infinite $\rho_k \in {}^*\mathbb{N}$ such that $\forall m_1, m_2 \in {}^*\mathbb{N}$ with $m(1/k) \leq m_1, m_2 \leq \rho_k$, we have $d(x_{m_1}, x_{m_2}) \leq \nu/k$. By Lemma 2.8, we can consider some infinite $\rho \in {}^*\mathbb{N}$ (in particular ρ is greater than any $m(r)$) such that $\rho \leq \rho_k$ for each $k \in \mathbb{N}^+$. Therefore, $\forall n \in \mathbb{N}^+ \ \forall m(1/n) \leq m' \in \mathbb{N} \ d([x_\rho], [x_{m'}]) \leq 1/n$, that is, $[x_\rho]$ is the limit of $\{[x_n]\}$.

(2) The $k-$bilipschitz map is defined by $[x] \mapsto [f(x)]$.

(3) Consider $[x], [y] \in C(X,p,\nu)$. Let $\gamma : [0,\lambda] \to {}^*X$ be a *geodesic from x to y. Then a geodesic $\delta : [0, st(\lambda/\nu)] \to C(X,p,\nu)$ from $[x]$ to $[y]$ is given by $\delta(t) = [\gamma(t\nu)]$.

(4) Consider $[x], [y] \in C(X,p,\nu)$. By hypothesis, there exists a *isometry $f : {}^*X \to {}^*Y$ such that $d(f(x), y) \leq C$ for some $C \in \mathbb{R}$. This *isometry induces a isometry of $C(X,p,\nu)$ in itself, which clearly maps $[x]$ to $[y]$.

□

Notice that point (2) implies in particular that *geodesics induce geodesics or geodesic rays or geodesic lines in appropriate asymptotic cones. Similarly for *quasi-geodesics. It is not difficult to show that the image of an induced geodesic or quasi-geodesic is the set induced by the image of the *geodesic or *quasi-geodesic, therefore it will still be harmless not to distinguish clearly between geodesics or quasi-geodesics and their images.

The usefulness of asymptotic cones relies on the fact that topological properties of asymptotic cones give quasi-isometric invariants, by property (2). Let us show an example of this. First, let us recall the topological characterizations of real trees (see [Be]).

Lemma 3.4. *The geodesic metric space X is a real tree if and only if for each pair of points $x, y \in X$ there is only one arc (i.e. injective path) joining them, up to reparametrization.*

It can be verified that each asymptotic cone of $\mathbb{H}^n$ (or, more in general, a simply connected, complete Riemannian manifold with sectional curvatures at most $-a$ for some $a > 0$) is a real tree. As an application, a compact flat manifold M of dimension $n \geq 2$ cannot have a hyperbolic metric, because otherwise by Milnor-Svarc Lemma we would have that $\mathbb{R}^n$ and $\mathbb{H}^n$ are quasi-isometric, by the fact that they are both quasi-isometric to the fundamental group of M. In particular there should be an asymptotic cone of $\mathbb{R}^n$ homeomorphic to an asymptotic cone of $\mathbb{H}^n$. But each asymptotic cone of $\mathbb{R}^n$ is $\mathbb{R}^n$, which is not a real tree.

Let us now introduce the asymptotic cones we are mostly interested in.

Definition 3.5. Let G be a finitely generated group and S a finite generating set for G. The asymptotic cone $C_S(G, g, \nu)$ of G with basepoint $g \in {}^*G$ and scaling factor $\nu \gg 1$ is $C(\mathcal{CG}_S(G), g, \nu)$.

From the properties of $\mathcal{CG}_S(G)$ and Lemma 3.3 we immediately obtain the following corollary.

Corollary 3.6. *For any finitely generated group G, finite generating sets S, S', $g, g' \in {}^*\mathbb{G}$, $\nu \gg 1$:*

- *$C_S(G, g, \nu)$ is complete, geodesic and homogeneous,*
- *$C_S(G, g, \nu)$ is isometric to $C_S(G, g', \nu)$,*
- *$C_S(G, g, \nu)$ is $k-$bilipschitz homeomorphic to $C_{S'}(G, g, \nu)$.*

In particular, notice that Lemma 3.3$-$(2) implies that topological properties of asymptotic cones do *not* depend on the choice of the finite generating system. These properties will therefore be of particular interest for us.

When a finite generating set S is fixed, we will often write $C(G, g, \nu)$ instead of $C_S(G, g, \nu)$.

3.1 Use of nonstandard methods for asymptotic cones

The main aim of this section is to show how nonstandard methods can be used to prove results about asymptotic cones.

We will see that the following lemma gives several obstructions for a space to be realized as an asymptotic cone.

Lemma 3.7. *An internal set is finite or has cardinality at least* $2^{\aleph_0}$.

Proof. Any set is finite or admits an injective function from $\mathbb{N}$. By the transfer of this property, we have that every internal set admits a bijective (internal) function from $\{0, \ldots, \nu\}$ for some $\nu \in {}^*\mathbb{N}$ or an injective (internal) function from ${}^*\mathbb{N}$.

So it is enough to prove that the set $\{0, \ldots, \nu\}$ is uncountable for every infinite ν. The fact that the map

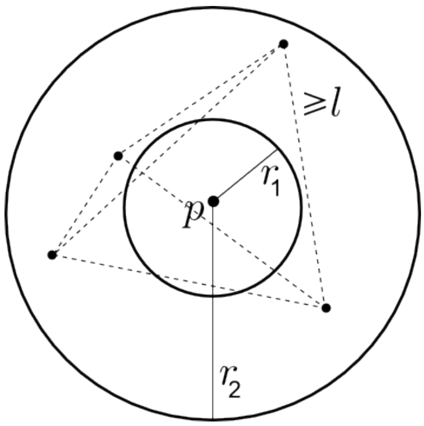
$$\alpha \in \{0, \ldots, \nu\} \mapsto st(\alpha/\nu) \in [0, 1]$$

is surjective implies the claim. $\square$

Let X be a metric space. For $p \in X$ and $r_1, r_2, l \geq 0$ denote by $F_X(p, r_1, r_2, l)$ the supremum of the cardinalities of sets M satisfying

1. $\forall x \in M,\ r_1 \leq d(x, p) \leq r_2$,
2. $\forall x, y \in M,\ x \neq y \Rightarrow d(x, y) \geq l$.

A set M satisfying the above properties will be called, for $\alpha \leq |M|$, a test for $F_X(p, r_1, r_2, l) \geq \alpha$.



Remark 3.8.

- If $F_X(p, r_1, r_2, l)$ is finite, then it is a maximum,
- for each $\alpha < F_X(p, r_1, r_2, l)$ we can find a test for $F_X(p, r_1, r_2, l) \geq \alpha$.

For convenience, if $\rho \in {}^*\mathbb{N}$ is infinite, we set $st(\rho) = 2^{\aleph_0}$. We also set $st(\alpha) = 2^{\aleph_0}$ for each *cardinality $\alpha \geq {}^*\aleph_0$.

We will often use the following easy properties:

Remark 3.9.

1. $F_X(p, r_1, r_2, l) \leq F_X(p', r_1', r_2', l')$ if and only if for each test M for $F_X(p, r_1, r_2, l) \geq \alpha$ there is a test M' for $F_X(p', r_1', r_2', l') \geq \alpha$,

2. if $r_1 \geq r'_1$, $r_2 \leq r'_2$ and $l \geq l'$ then $F_X(p, r_1, r_2, l) \leq F_X(p, r'_1, r'_2, l')$,

3. if X is the asymptotic cone of Y with basepoint p and scaling factor ν, then $st\,(F_{^*Y}(p, \rho_1\nu, \rho_2\nu, \lambda\nu)) \leq F_X([p], st(\rho_1), st(\rho_2), st(\lambda))$, where ρ_1, ρ_2 and λ are finite and $st(\lambda) > 0$.

Proof. (1) is straightforward from the definitions.

(2) If M is a test for $F_X(p, r_1, r_2, l) \geq \alpha$, then it is also a test for $F_X(p, r'_1, r'_2, l') \geq \alpha$.

(3) By our convention on the standard part and the fact that $|^*\mathbb{N}| = 2^{\aleph_0}$, $st(\nu)$ is the cardinality of $\nu = \{0, \ldots, \nu - 1\}$. In fact, if ν is finite, $st(\nu) = \nu$, otherwise $st(\nu) = 2^{\aleph_0} \leq |\nu| \leq |^*\mathbb{N}| = 2^{\aleph_0}$ (we used Lemma 3.7). If M is a test for $F_{^*Y}(p, \rho_1\nu, \rho_2\nu, \lambda\nu) \geq \alpha$, its projection on X is a test for $F_X([p], st(\rho_1), st(\rho_2), st(\lambda)) \geq st(\alpha)$ (projections of distinct elements of M are distinct because their distance is at least $st(\lambda) > 0$).

$\square$

Proposition 3.10. *Let X be a metric space. If X is an asymptotic cone then for each p, r_1, r_2, l, with $l > 0$, if $F_X(p, r_1, r_2, l)$ is infinite then it is at least $2^{\aleph_0}$.*

Proof. Assume that X is an asymptotic cone of Y, with scaling factor ν, and fix p, r_1, r_2, l as above and such that $F_X(p, r_1, r_2, l) \geq \aleph_0$. Fix a representative $\pi \in {^*Y}$ of p. For each n, one can find $x_{n,1}, \ldots, x_{n,n} \in {^*Y}$ such that

- each $x_{n,i}$ is at a distance $(r_{n,i} + \xi_{n,i})\nu$ from π, for some $r \in [r_1, r_2]$ and some infinitesimal $\xi_{n,i}$,
- for each n and $i \neq j$, $d(x_{n,i}, x_{n,j}) > (l - \xi_{n,i,j})\nu$ for some positive infinitesimal $\xi_{n,i,j}$.

We can bound all the $|\xi_{n,i}|$'s and $\xi_{n,i,j}$ by some positive infinitesimal ξ, by Lemma 2.8. So we have that $F_{^*Y}(\pi, (r_1 - \xi)\nu, (r_2 + \xi)\nu, (l - \xi)\nu)$ is greater than any finite n, hence it is greater than some infinite $\rho \in {^*\mathbb{N}}$. The conclusion follows from point (3) of Remark 3.9.

$\square$

We will now study the consequences of this proposition for real trees appearing as asymptotic cones. These are interesting objects in view of the fact that a geodesic metric space is hyperbolic if and only if each of its asymptotic cones is a real tree (see [Dr1] or [FS]). Groups such that at least one of their asymptotic cones is a real tree are studied in [OOS].

Corollary 3.11. *If X is a real tree such that every geodesic can be extended (e.g.: a homogeneous real tree, see [DP]) and the valency at a point p is infinite, then this valency is at least $2^{\aleph_0}$.*

Proof. Our assumption on geodesics implies that, for each $r > 0$, $F_X(p, r, r, 2r)$ equals the valency at p.

$\square$

Definition 3.12. In a real tree, a point of valency greater than 2 will be called a branching point.

Proposition 3.13. *Let X be a real tree such that each geodesic can be extended and the valency at a point p is finite. If X is an asymptotic cone then p is isolated from the other branching points.*

Proof. Let n be the valency of X at p. For each $r > 0$, $F_X(p, r, r, 2r) = n$. Assume that p is not isolated from the other branching points. Then for each $k \in \mathbb{N}$ (and $k > 1/2r$) we have that $F_X(p, r, r, 2r - 1/k)$ is infinite. If X is an asymptotic cone of Y with scaling factor ν and $\pi \in {}^*Y$ is a representative for p, proceeding as in the proof of Proposition 3.10, for each k we can find a positive infinitesimal ξ_k and a positive infinite μ_k such that $F_{^*Y}(\pi, (r - \xi_k)\nu, (r + \xi_k)\nu, (2r - 1/k - \xi_k)\nu) \geq \mu_k$. Let us fix a positive infinitesimal ξ greater than any ξ_k and a positive infinite μ smaller than any μ_k. We have that $\{\alpha | F_{^*Y}(\pi, (r - \xi)\nu, (r + \xi)\nu, (2r - \alpha)\nu) \geq \mu\}$ contains, for each k, elements of ${}^*\mathbb{R}$ smaller than $1/k$ (for example $1/(k+1) + \xi_{k+1}$), hence it contains an infinitesimal η. This implies that $F_X(p, r, r, 2r)$ is infinite (using point (3) of Remark 3.9), a contradiction.

$\square$

Putting together Corollary 3.11 and Proposition 3.13 in the case of homogeneous real trees, we have:

Corollary 3.14. *If X is a homogeneous real tree and an asymptotic cone, then it is a point, a line or it has valency at least $2^{\aleph_0}$ at each point.*

As finitely generated groups are countable (from now on, groups are implied to be finitely generated), their asymptotic cones have cardinality at most $2^{\aleph_0}$ and hence:

Corollary 3.15. *If the homogeneous real tree X is the asymptotic cone of a group, then it is a point, a line or a tree with valency $2^{\aleph_0}$ at each point.*

In view of the following theorem, proved in [DP], this corollary imply that there are 3 possible isometries types of real trees appearing as asymptotic cones of groups.

Theorem 3.16. *If T_1, T_2 are homogeneous real trees such that the valency at a point in T_1 is the same as the valency at a point in T_2, then T_1 and T_2 are isometric.*

Now, let us analyze the consequences of Proposition 3.10 in the special case $r_1 = 0$.

Theorem 3.17. *1. If the separable metric space X is an asymptotic cone, then X is proper.*

2. Suppose that for some infinite ν each asymptotic cone of the metric space Y with scaling factor ν is separable. Then all those asymptotic cones have finite Hausdorff dimension.

Proof. (1) Let X be the asymptotic cone of Y with basepoint p and scaling factor ν. Suppose that $B = \overline{B}([p], r) \subseteq X$ is not compact. Then, as X and therefore B is complete, there exists some $0 < \epsilon < 1$ such that B cannot be covered by finitely many balls of radius ϵr. Set $r' = \epsilon r/2$. Suppose by contradiction that $F_{^*Y}(p, 0, (r+1)\nu, r'\nu)$ is finite (say equal to n), consider a test $M \subseteq {}^*Y$ of $F_{^*Y}(p, 0, (r+1)\nu, r'\nu) \geq n$. Let $\{x_1, \dots, x_n\} \subseteq X$ be the projection of M onto X (the x_i's are distinct because $r' > 0$). As, by our hypothesis, B cannot be a subset of $\bigcup B(x_i, \epsilon r)$, we can find $[y] \in B \backslash \bigcup B(x_i, \epsilon r)$. As $r' < \epsilon r$, it is easily seen that $M \cup \{y\}$ is a test for $F_{^*Y}(p, 0, (r+1)\nu, r'\nu) \geq n+1$, which is a contradiction: in fact, $d(p, y) < (r+1)\nu$ as $st(d(p, y)/\nu) = r$, and, for each $m \in M$, $d(y, m) \geq r'\nu$ as $st(d(p, m)/\nu) \geq \epsilon r > r'$.

Therefore, $F_{^*Y}(p, 0, (r+1)\nu, r'\nu) \geq \rho$, for some infinite $\rho \in {}^*\mathbb{N}$. Let us show that this implies that $B([p], r+1)$ is not separable, as it should be as X is separable (subsets of separable metric spaces are separable). We have that there exists a test M for $F_X([p], 0, r+1, r') \geq 2^{\aleph_0}$, which is obtained projecting onto X a test for $F_{^*Y}(p, 0, (r+1)\nu, r'\nu) \geq \rho$. If $m_1, m_2 \in M$ are distinct, we have $B(m_1, r'/2) \cap B(m_2, r'/2) = \emptyset$ as $d(m_1, m_2) \geq r'$. This gives an uncountable family of non-intersecting balls whose centers lie in $B([p], r+1)$. The existence of such a family clearly implies that $B([p], r+1)$ is not a separable space.

(2) We will prove that there exists $n \in \mathbb{N}$ such that for each $r > 0$ and each asymptotic cone X of Y with scaling factor ν, any ball of radius r in X can be covered by at most n balls of radius $r/2$.

Suppose that for some $p \in {}^*Y$ we had that $F_{^*Y}(p, 0, (r+1)\nu, r\nu/4) \geq \rho$ for some infinite $\rho \in {}^*\mathbb{N}$ and let X be the asymptotic cone of Y with basepoint p and scaling factor ν. The final argument from the proof of point (1) would show that $B([p], r+1) \subseteq X$ is not separable, therefore

$$A = \{\nu \in {}^*\mathbb{N} : \exists p \in {}^*Y \; F_{^*Y}(p, 0, (r+1)\nu, r\nu/4) \geq \nu\}$$

is contained in $\mathbb{N} \subseteq {}^*\mathbb{N}$. As it is internal, it must be a finite set. Therefore, each element of A is bounded by, say, $n \in \mathbb{N}$. Consider some ball $B([p], r)$ in an asymptotic cone X with scaling factor ν, and set $F_{^*Y}(p, 0, (r+1)\nu, r\nu/4) = m \leq n$. Let M be a test for $F_{^*Y}(p, 0, (r+1)\nu, r\nu/4) \geq m$ (in particular $|M| = m$). Let $\{x_1, \dots, x_m\}$ be the projection of M onto X (as usual, the x_i's are distinct). We have that $B([p], r) \subseteq \bigcup B(x_i, r/2)$, for otherwise we could construct a test for $F_{^*Y}(p, 0, (r+1)\nu, r\nu/4) \geq m+1$, as we did in the proof of point (1).

□

Corollary 3.18. *If the group G has one separable asymptotic cone, then it is virtually nilpotent. The same is true by substituting "separable" with "locally compact".*

Proof. If an asymptotic cone Y of G is separable, all the asymptotic cones with the same scaling factor are separable, being homeomorphic to Y. Therefore, we can apply both points of the theorem above to obtain that Y is locally compact and finite dimensional. In the proof of the theorem by Gromov about groups of polynomial growth, only the following properties of an asymptotic cone Y of such a group are used in order to prove that it is virtually nilpotent (as remarked in [vDW, 1.11]):

1. Y is homogeneous,
2. Y is connected and locally connected,
3. Y is complete,
4. Y is locally compact and finite dimensional.

(3) is true for any asymptotic cone, (2) is true for any asymptotic cone of a geodesic space (as such an asymptotic cone is geodesic), (1) holds for any asymptotic cone of a group, as groups are quasi-homogeneous. Finally, we proved (4) in the hypothesis of separability. This completes the proof of the first part of the corollary. On the other hand, local compactness and completeness in a geodesic metric space imply properness (see Proposition I.3.7 in [BH]), and hence separability.

□

Remark 3.19. The corollary above answers the question (asked in [vDW, Remark 6.4-(3)]) whether or not local compactness of one asymptotic cone of a group implies that the group is of polynomial growth, even if the proof is far from being direct.

For completeness, let us state the following result about the structure of asymptotic cones of virtually nilpotent groups.

Theorem 3.20. *Consider a virtually nilpotent group G, and fix a finite generating set for G. All the asymptotic cones of G are isometric, and isometric to a Lie group G_∞ endowed with a left invariant Carnot-Carathéodory metric d_∞ (see [Pa] for the description of G_∞ and d_∞). The Hausdorff dimension of G_∞ is the rate of growth of G.*

Corollary 3.21. *If the group G has one separable or locally compact asymptotic cone then* each *asymptotic cone of G is a manifold.*

Proof. By Corollary 3.18, G is virtually nilpotent, so the theorem above applies. $\square$

As pointed out in [Dr1], the theorem above follows from the results in [Pa], together with the fact that Gromov-Hausdorff convergence of a sequence of pointed proper metric spaces $\{(X_i, x_i)\}$ to some space X_∞ implies that all the ultralimits of $\{(X_i, x_i)\}$ are isometric to X (which is an easy consequence of [KL, Proposition 3.2]). Let us restate this last fact in the nonstandard language.

If (X, x_0) is a pointed *metric space, define the equivalence relation $\sim$ on X by $x \sim y \iff d(x,y) \ll 1$. Set $\overline{X} = \{[x] \in X/\sim : d(x, x_0) \in O(1)\}$, and define a distance on $\overline{X}$ by $d([x],[y]) = st(d_X(x,y))$ (this is a well-known procedure in nonstandard analysis which enables to embed any metric space in a complete metric space, see [Go]). The restatement of the fact above is: if $\{(X_i, x_i)\}$ is a sequence of pointed proper metric spaces which converges in the Gromov-Hausdorff sense to X_∞, then for each infinite $\rho \in {}^*\mathbb{N}$, we have that $\overline{X_\rho}$ is isometric to X_∞.

As it is fairly easy, let us prove a simpler statement than that of the theorem.

Proposition 3.22. *Each asymptotic cone of a virtually nilpotent group is proper and has finite Hausdorff dimension.*

Proof. Let G be a virtually nilpotent group, and fix a finite generating set S. Set, for $n \in \mathbb{N}$, $\gamma(n) = |\{g \in G : d(e,g) \leq n\}|$. By [Ba, Theorem 2], there exists $d = d(G)$ and constants $A, B > 0$ such that, for each $n \in \mathbb{N}^+$,

$$An^d \leq \gamma(n) \leq Bn^d.$$

We will need both bounds. Set $X = \mathcal{CG}_S(G)$. We want to show that, for each $r \in \mathbb{R}^+$ with $r \geq 100$ (for convenience), there can be at most K disjoint balls of radius $r/4$ in a ball of radius r, for some K which does not depend on r. Once we do so, it is quite clear that each ball B of radius l in an asymptotic cone Y of G can be covered by at most K balls of radius $l/2$ (for example, by arguments similar to those of the proof of Theorem 3.17). This implies that Y has finite Hausdorff dimension, but also that it is proper (as Y is complete).

Fix $r \in \mathbb{R}^+$, $r \geq 100$, and set $n = \lfloor r/4 - 1/2 \rfloor$, $N = \lfloor r \rfloor + 1$ (where $\lfloor x \rfloor$ denotes the integer part). Any ball of radius $r/4$ in X contains a ball centered in an element of G of radius n. Any such ball of radius n contains at least An^d elements of G, while $\overline{B}_N(e) \supseteq \overline{B}_r(e)$ contains at most BN^d points.

These considerations readily imply that there can be at most

$$\frac{BN^d}{An^d} \leq \frac{B}{A}\left(\frac{r+1}{r/4 - 3/2}\right)^d \leq 5^d B/A$$

disjoint balls of radius $r/4$ contained in $\overline{B}_r(e)$.

$\square$

Remark 3.23. Refining the proof it is possible to show that the Hausdorff dimensions of the asymptotic cones can be bounded by the growth rate of the group. Along the same lines, one can show that these Hausdorff dimensions coincide with the growth rate.

Theorem 3.17 provides many examples of metric spaces which do not appear as asymptotic cones, for example the separable Hilbert space. In contrast, we will prove below a "positive" result on spaces which are realized as asymptotic cones.

Theorem 3.24. *If the metric space X is proper, then it is an asymptotic cone of some metric space Y. If X is also geodesic and unbounded, we can choose Y to be geodesic as well.*

The construction in the proof is a slight variation of the one which appears in [FS] (Section 5), translated in the nonstandard setting.

Proof. Let us first assume that X is unbounded. Let $\{p_n\}$ be a sequence of points of X such that $d(p_0, p_n) \to \infty$. Set $Y = (X \times \mathbb{N}) \cup \bigcup_{n\in\mathbb{N}}(\{p_n\} \times [n, n+1])$. Define a distance $\widetilde{d}$ on Y in the following way:

$$\widetilde{d}((x,t),(x',t')) = \begin{cases} t \cdot d(x, p_t) + t' \cdot d(p_{t'}, x') + |t - t'| & \text{if} \quad t \neq t' \\ t \cdot d(x, x') & \text{if} \quad t = t' \end{cases}$$

It is quite clear that Y is a metric space, and that it is geodesic if X is geodesic. Consider now ${}^*Y = ({}^*X \times {}^*\mathbb{N}) \cup \bigcup_{\mu\in {}^*\mathbb{N}}(\{p_\mu\} \times [\mu, \mu+1])$, and an infinite $\nu \in {}^*\mathbb{N}$. We want to show that the asymptotic cone Z of Y with basepoint (p_0, ν) and scaling factor ν is isometric to X. The isometry $i : X \to Z$ can be defined simply by $x \mapsto [(x, \nu)]$. It is readily checked that it is an isometric embedding. So far we did not use properness or that $d(p_0, p_n) \to \infty$, so we obtained the following.

Remark 3.25. Any metric space X can be isometrically embedded in an asymptotic cone of a metric space Y. If X is geodesic, we can require Y to be geodesic.

Section 5 of [FS] already contains a proof of this fact.

We are left to prove that i is surjective. First of all, notice that the distance of any element of ${}^*Y \backslash ({}^*X \times \{\nu\})$ from (p_0, ν) is at least

$$\min\{\nu d(p_0, p_{\nu-1}), \nu d(p_0, p_{\nu+1})\} \gg \nu,$$

as $d(p_0, p_n) \to \infty$ and so $d(p_0, p_\mu) \gg 1$ for each infinite $\mu \in {}^*\mathbb{N}$. Therefore no element of ${}^*Y \backslash ({}^*X \times \{\nu\})$ projects onto an element of Z. What remains to prove is that for each $y \in {}^*X$ with $\widetilde{d}((p_0, \nu), (y, \nu))/\nu = d(p_0, y) \in O(1)$

there exists $x \in X$ such that $\widetilde{d}((x,\nu),(y,\nu))/\nu = d(x,y) \ll 1$. Consider y as above and some $r > d(p_0,y)$, $r \in \mathbb{R}$. We have that $B = \overline{B}_X(p_0,r)$ is compact and $y \in {}^*B$. By the nonstandard characterization of compact metric spaces (Proposition 2.21), there exists $x \in B$ such that $d(x,y) \ll 1$, and we are done.

The case that X is bounded can be handled similarly. Fix $p \in X$ and set $Y = X \times \mathbb{N}$. Define

$$\widetilde{d}((x,n),(x',n')) = \begin{cases} n \cdot d(x,p) + n' \cdot d(p,x') + |n^2 - (n')^2| & \text{if} \quad n \neq n' \\ n \cdot d(x,x') & \text{if} \quad n = n' \end{cases}$$

Modifying the previous proof, it is easily shown that, for any infinite $\nu \in {}^*\mathbb{N}$, the asymptotic cone of Y with basepoint (p,ν) and scaling factor ν is isometric to X.

□

Let us change topic and define a weaker notion than that of being quasi-isometrically embedded.

Definition 3.26. The pointed metric space (X,x) can be weakly $k-$bilipschitz embedded in the metric space Y if there exists a function $C : \mathbb{R}^+ \to \mathbb{R}^+$ such that

- $\lim_{r\to+\infty} C(r)/r = 0$,
- for each $r \in \mathbb{R}^+$, $B_r(x)$ can be $(k, C(r))-$quasi-isometrically embedded in Y.

If (X,x) can be weakly $k-$bilipschitz embedded in Y for each $x \in X$, we will say that X can be weakly $k-$bilipschitz embedded in Y.

Proposition 3.27. *Consider a pointed metric space (X,x) such that each asymptotic cone of X with basepoint $x \in X$ is separable. Let Y be a metric space. Assume that there exists $k > 0$ such that for each scaling factor ν, there exists $y_\nu \in {}^*Y$ such that $C(X,x,\nu)$ can be weakly $k-$bilipschitz embedded in $C(Y,y_\nu,\nu)$. Then (X,x) can be weakly $k-$bilipschitz embedded in Y.*

Proof. Fix some infinite ν. For short, let us denote $C(X,x,\nu)$ by C_X and $C(Y,y_\nu,\nu)$ by C_Y. The closed ball B in C_X of radius 1 centered in $[x]$ is compact, by Theorem 3.17, and therefore for each $n \in \mathbb{N}^+$ there exists a finite set $\{[x_n^1],\dots,[x_n^{j(n)}]\}$ which is $1/n-$dense in B. Notice that there exists an infinitesimal ξ_n such that for each point $a \in B_{^*X}(x,\nu)$ there is a point in $X_n = \{x_n^1,\dots,x_n^{j(n)}\}$ closer than $(1/n+\xi_n)\nu$ to a. Let $\phi : C_X \to C_Y$ be a $k-$bilipschitz embedding. Fix some $n \in \mathbb{N}^+$. For each i, choose an element $f(x_n^i)$ in ${}^*Y$ which projects on $\phi([x_n^i])$. There exists an infinitesimal η_n such that $f : X_n \to {}^*Y$ is a $(k,\eta_n\nu)-$quasi-isometric embedding. Choose now an internal map $\psi : B_{^*X}(x,\nu) \to X_n$ such that $d(\psi(z), z) \leq (1/n + \xi_n)\nu$ for each $z \in B_{^*X}(x,\nu)$. Such map exists by the nonstandard interpretation of the Axiom of Choice. We have that ψ is an internal $(1, 2(1/n + \xi_n)\nu)-$quasi-isometric embedding. Consider the map $\Phi_n : B_{^*X}(x,\nu) \to {^*Y}$ defined by $\Phi(z) = f(\psi(z))$. We have that Φ is an internal $(k, ((1/n + \xi_n)2k + \eta_n)\nu)-$quasi-isometric embedding, as it is a composition of internal quasi-isometric embeddings with constants $(1, 2(1/n + \xi_n)\nu)$ and $(k, \eta_n\nu)$.

The discussion above shows that the set of internal $(k, \nu/m)-$quasi-isometric embeddings of $B_{^*X}(x,\nu)$ in ${^*Y}$ is not empty for each $m \in \mathbb{N}^+$ (for $n > 2km$ we have that $(1/n + \eta_n)2k + \eta_n < 1/m$, as this is true taking the standard parts on both sides). As this holds for each infinite ν, given any $m \in \mathbb{N}^+$ there exists $r_0 \in \mathbb{R}^+$ such that

$$\forall r \geq r_0 \; \exists f \in QI_{k,m}(B_{^*X}(x,r), {^*Y}),$$

where $QI_{k,m}(B_{^*X}(x,r), {^*Y})$ is the internal set of all internal $(k, r/m)$-quasi-isometric embeddings from $B_{^*X}(x,r)$ to ${^*Y}$. By the transfer principle (used "backwards"), this implies that for each $r \geq r_0$ there exists a $(k, r/m)-$quasi-isometric embedding from $B_X(x,r)$ to Y. □

Let us apply this result to the case of asymptotic cones of virtually nilpotent groups.

Corollary 3.28. *If G and H are virtually nilpotent groups, with fixed generating systems, and, for each infinite ν, $C(G,\nu)$ can be $k-$bilipschitz embedded in $C(H,\nu)$, then G can be weakly $k-$bilipschitz embedded in H.*

Remark 3.29. It is possible that in the same hypotheses one can conclude that G can be quasi-isometrically embedded in H.

Chapter 4

Tree-graded spaces

The asymptotic cones we will be interested in have the structure which we will describe in this chapter. All results and definitions before Lemma 4.9 are taken from [DS].

Definition 4.1. A geodesic complete metric space $\mathbb{F}$ is tree-graded with respect to a collection $\mathcal{P}$ of closed geodesic subsets of $\mathbb{F}$ (called *pieces*) if the following properties are satisfied:

(T_1) two different pieces intersect in at most one point,

(T_2) each geodesic simple triangle is contained in one piece.

Also, pieces are required to cover $\mathbb{F}$ (which follows from (T_2) if we set the convention that trivial triangles are simple).

Throughout the chapter, let $\mathbb{F}$ denote a tree-graded space with respect to $\mathcal{P}$. Notice that if each $P \in \mathcal{P}$ is a real tree, then $\mathbb{F}$ is a real tree as well.

Notice that the definition of tree-graded space is given in terms of its metric, not just its topology (we are interested in topological properties of tree-graded spaces, as they will appear as asymptotic cones). However, it turns out that one can deduce many topological properties. For example, here is the topological analogue of property (T_2):

Lemma 4.2. *Each simple* loop *in $\mathbb{F}$ is contained in one piece.*

In particular, for example, simple quadrangles are contained in one piece.

The most powerful technical tool for studying tree-graded spaces are the projections defined in the following lemma.

Lemma 4.3. *For each $P \in \mathcal{P}$ there exists a map $\pi_P : \mathbb{F} \to P$, called the projection on P, such that for each $x \in \mathbb{F}$:*

- *$d(x, P) = d(x, \pi_P(x))$,*
- *each curve (in particular each geodesic) from x to a point in P contains $\pi_P(x)$,*

- *π_P is locally constant outside $\mathbb{F}$. In particular, if $A \subseteq \mathbb{F}$ ($A \neq \emptyset$) is connected and $|A \cap P| \leq 1$, $\pi_P(A)$ consists of one point.*

Notice that if $x \in P$, then $\pi_P(x) = x$. As an example of use of projections, let us derive 3 statements we will need.

Corollary 4.4. *Each arc (i.e. injective path) connecting 2 points of a piece P is contained in P. In particular the intersection between a geodesic and a piece is either empty, a point or a subgeodesic.*

Proof. Let γ be an arc as in the statement. Suppose that there is a point p on γ outside P and let γ' be the subpath which contains p and intersects P only in its endpoints x, y. Notice that $x \neq y$, because γ is injective. As $A = \gamma' \backslash \{x\}$ is connected and it intersects P only in y, $\pi_P(A) = \pi_P(y) = y$. But, for a similar reason, we should have $\pi_P(A) = x \neq y$, a contradiction. □

Corollary 4.5. *A geodesic ray $\gamma : [0, \infty) \to \mathbb{F}$ which stays at bounded distance from a piece P is definitively contained in P.*

Proof. If the geodesic ray γ does not intersect P (or intersects it only in one point), π_P is constant along γ. Therefore $d(\gamma(t), P) = d(\gamma(t), \pi_P(\gamma(t)))$ is not bounded. This readily implies that there are arbitrarily large t's such that $\gamma(t) \in P$. We conclude using the previous corollary. □

Corollary 4.6. *If x, y are such that $\pi_P(x) \neq \pi_P(y)$, for some piece P, then any geodesic δ from x to y intersects P.*

Proof. If we had $\delta \cap P = \emptyset$, then π_P would be constant along δ and so $\pi_P(x) = \pi_P(y)$.

□

Another concept which will turn out to be useful is that of transversal tree:

Definition 4.7. For each $x \in \mathbb{F}$ denote by T_x the set of points $y \in \mathbb{F}$ such that there exists a path joining x to y which intersects each piece in at most one point.

Basic properties of transversal trees are given below.

Lemma 4.8. *For each $x \in \mathbb{F}$*

- *T_x is a real tree,*
- *T_x is closed in $\mathbb{F}$,*
- *if $y \in T_x$, then $T_x = T_y$,*
- *every arc joining $y, x \in T_x$ is contained in T_x.*

The reader may visualize $\mathbb{F}$ as a space in which paths are concatenations of paths contained in a piece and paths contained in a transversal tree (this is not quite true, but it helps to visualize $\mathbb{F}$) and if two paths leave some piece in 2 different points x, y, then they start to move in far away regions of $\mathbb{F}$ meaning that a path joining these two regions pass through x and y.

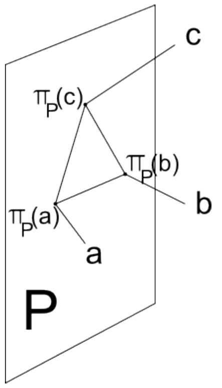


Figure 4.1: A geodesic triangle in a tree-graded space

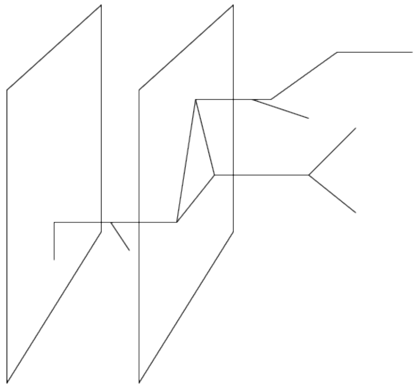


Figure 4.2: Geodesics in a tree-graded space.

The next lemma will be used a lot of times.

If β and γ are geodesics such that the final point of β is the initial point of γ, we will denote their concatenation by $\beta\gamma$.

Lemma 4.9. *Suppose that γ_1 and γ_2 are geodesics or geodesic rays in a tree-graded space such that*

1. *the final point p of γ_1 is the starting point of γ_2,*
2. *$\gamma_1 \cap \gamma_2 = \{p\}$,*
3. *there is no piece containing a final subpath of γ_1 and an initial subpath of γ_2.*

Then $\gamma_1\gamma_2$ is a geodesic (or a geodesic ray or a geodesic line). Also, each geodesic from a point in γ_1 to a point in γ_2 contains p.

Proof. If the thesis were false, we would have points $q \in \gamma_1$, $r \in \gamma_2$ such that $d(q,r) < d(q,p) + d(p,r)$. Consider a geodesic triangle with vertices p, q, r and $[q,p] \subseteq \gamma_1$, $[p,r] \subseteq \gamma_2$. Condition (2) and $d(q,r) < d(q,p)+d(p,r)$ imply that it cannot be a tripod, for otherwise $[q,p] \cap [p,q]$ should contain a non-trivial geodesic. Therefore there exists a piece P intersecting both $[q,p]$ and $[p,r]$. Condition (3) implies that P does not contain p. But then both $[q,p]$ and $[p,r]$ should pass through $\pi_P(p) \neq p$, which contradicts (2).

The last part of the statement has a similar proof.

□

We point out that using the lemma above we can give a complete criterion for whether or not the composition of 2 geodesics in a tree-graded space is a geodesic. This criterion is somehow "local" in the concatenation point.

Remark 4.10. The concatenation of geodesics γ_1, γ_2 in a tree-graded space is a geodesic if and only if conditions $(1), (2)$ and (3) or $(1), (2)$ and $(3')$ hold, where $(3')$ is

$(3')$ there is a piece P such that $\gamma_i \cap P = \beta_i$ are non-trivial subgeodesics containing the final point of γ_1 and $\beta_1\beta_2$ is a geodesic in P.

Definition 4.11. We will say that γ_1 and γ_2 concatenate well if they satisfy the hypothesis of Lemma 4.9.

Remark 4.12. Suppose that γ_2 and γ_2' are geodesics starting from a certain point p such that γ_2' and γ_2 concatenate well. Also, suppose that γ_1 is a geodesic whose final point is p. Then either γ_1 and γ_2 or γ_1 and γ_2' concatenate well.

4.1 Alternative definition

In this section we give a characterization of tree-graded spaces, which will be convenient to prove that certain spaces are tree-graded.

Throughout the section, let us denote by X a complete geodesic metric space and by $\mathcal{P}$ a collection of closed geodesic subsets of X, which cover X. We want to capture the fundamental properties of projections on a piece in a tree-graded space.

Definition 4.13. A family of maps $\Pi = \{\pi_P : X \to P\}_{P\in\mathcal{P}}$ will be called projection system for $\mathcal{P}$ if, for each $P \in \mathcal{P}$,

$(P1)$ for each $r \in P$, $z \in X$, $d(r, z) = d(r, \pi_P(z)) + d(\pi_P(z), z)$,

$(P2)$ π_P is locally constant outside P,

$(P3)$ for each $Q \in \mathcal{P}$ with $P \neq Q$, we have that $\pi_P(Q)$ is a point.

Remark 4.14. Notice that $\pi_P(x)$ is a point which minimizes the distance from x to P. In particular, if $x \in P$, then $\pi_P(x) = x$.

Lemma 4.15. *Suppose that $\{\pi_P\}$ is a projection system.*

1. *Consider $x \in X$ and $P \in \mathcal{P}$. Each arc (in particular, each geodesic) from x to some $p \in P$ passes through $\pi_P(x)$.*

2. *For each $P \in \mathcal{P}$ and each arc (in particular, each geodesic) connecting 2 points on P in contained in P. As a consequence, the intersection between an arc γ and $P \in \mathcal{P}$ is either empty, a point or a subarc of γ.*

3. *each simple loop which intersects some $P \in \mathcal{P}$ in more than one point is contained in P.*

Proof. (1) Consider an arc $\gamma : [0,t] \to \mathbb{F}$ from x to p. Let $q = \gamma(u)$ be the first point of $\gamma \cap P$ (P is closed in X by assumption). By $(P2)$, $\pi_P \circ \gamma|_{[0,u)}$ is constant, so $\pi_P(x) = \pi_P(\gamma(u'))$ for each $u' \in [0,u)$. Using this fact and $(P1)$ with $r = q$ and $z = \gamma(u')$, for $u' \in [0,u)$, we get

$$d(q, \gamma(u')) = d(q, \pi_P(x)) + d(\pi_P(x), \gamma(u')).$$

As u' tends to u, the left-hand side tends to $d(q,q) = 0$, while the right-hand side tends to $2d(q, \pi_P(x))$. Therefore $d(q, \pi_P(x)) = 0$ and $q = \pi_P(x)$.

(2) Consider an arc γ between two points in some $P \in \mathcal{P}$ and suppose by contradiction that there exists $x \in (\gamma \backslash P)$. We can consider a subarc γ' of γ containing x and with endpoints $x_1 \neq x_2$ with the property that $\gamma' \cap P = \{x_1, x_2\}$. We have that $[x, x_1]$ intersects P only in its endpoint. By what we proved so far, we must have $\pi_P(x) = x_1$. But, for the very same reason, we should also have $\pi_P(x) = x_2$, a contradiction.

(3) The loop as in the statement can be considered as the union of 2 arcs connecting points on P. The conclusion follows from point (2).

□

The characterization of projection systems given below will be helpful for future aims.

Lemma 4.16. *Properties $(P1)$ and $(P2)$ can be substituted by:*
$(P'1)$ for each $P \in \mathcal{P}$ and $x \in P$, $\pi_P(x) = x$,
$(P'2)$ for each $P \in \mathcal{P}$ and for each $z_1, z_2 \in X$ such that $\pi_P(z_1) \neq \pi_P(z_2)$,

$$d(z_1, z_2) = d(z_1, \pi_P(z_1)) + d(\pi_P(z_1), \pi_P(z_2)) + d(\pi_P(z_2), z_2).$$

Proof. Assume that $\{\pi_P\}$ satisfies $(P'1)$ and $(P'2)$. Property $(P1)$ is not trivial only if $r \neq \pi_P(z)$, and in this case follows from $(P'2)$ setting $z_1 = z$, $z_2 = r$ and taking into account that, by $(P'1)$, $\pi_P(r) = r \neq \pi_P(z)$. As we have property $(P1)$, we also have that $d(z, P) = d(z, \pi_P(z))$ for each $z \in X$. Hence, property $(P2)$ follows from the fact that if $\pi_P(z_1) \neq \pi_P(z_2)$ then $d(z_1, z_2) > d(z_1, P)$.

Assume that $\{\pi_P\}$ satisfies $(P1)$ and $(P2)$. We already remarked that $(P'1)$ holds. Consider z_1, z_2, P as in property $(P'2)$, and a geodesic δ between z_1 and z_2. If we had $\delta \cap P = \emptyset$, then π_P would be constant along δ and so $\pi_P(z_1) = \pi_P(z_2)$. Therefore δ intersects P. So, by point (1) of the previous lemma, δ contains $\pi_P(z_1)$ and $\pi_P(z_2)$, hence the thesis.

□

Definition 4.17. A geodesic is $\mathcal{P}-$transverse if it intersects each $P \in \mathcal{P}$ in at most one point. A geodesic triangle in X is $\mathcal{P}-$transverse if each side is $\mathcal{P}-$transverse.

$\mathcal{P}$ is transverse-free if each $\mathcal{P}-$transverse geodesic triangle is a tripod.

Theorem 4.18. *X is tree-graded with respect to $\mathcal{P}$ if and only if $\mathcal{P}$ is transverse-free and there exists a projection system for $\mathcal{P}$.*

Remark 4.19. The request for $\mathcal{P}$ to be transverse-free guarantees that $\mathcal{P}$ contains "enough" subspaces of X.

Proof. $\Rightarrow$: This implication follows from the properties of tree-graded spaces we stated before.

$\Leftarrow$: Let $\Pi = \{\pi_P\}$ be a projection system for P. Let us prove property (T_1). Consider $P, Q \in \mathcal{P}$ with $P \neq Q$. If $x, y \in P \cap Q$, we have $\pi_P(Q) \supseteq \{x, y\}$. By $(P3)$, this implies $x = y$.

Let us show how to obtain property (T_2). Consider a simple geodesic triangle Δ with vertices a, b, c. If it consists of one point (recall that we consider these triangles to be simple), then it is contained in some $P \in \mathcal{P}$, as we assume that elements of $\mathcal{P}$ cover X. So, we can suppose that Δ is not trivial. Then, it cannot be $\mathcal{P}-$transverse, for otherwise it would be a non-trivial tripod, and therefore not a simple triangle.

So, we can assume that $P \cap [a, b]$ contains a non-trivial subgeodesic $[a', b'] \subseteq [a, b]$, for some $P \in \mathcal{P}$. The conclusion follows from Lemma 4.15$-$(3), as Δ is in particular a simple loop.

□

Remark 4.20. Property $(P3)$ was used only to prove (T_1). Therefore, another way to prove that X is tree-graded is to prove property (T_1), properties $(P1)$ and $(P2)$ (or $(P'1)$ and $(P'2)$) for some family of maps $\{\pi_P\}$, and that $\mathcal{P}$ is transverse-free.

Typically, it is hard to have control on each geodesic of a metric space, therefore it can be hard to prove that $\mathcal{P}$ is transverse-free. However, the following lemma tells us that it is enough to know, for each pair of point, one geodesic connecting them.

Lemma 4.21. *Suppose that there exists a projection system for $\mathcal{P}$. Consider points $p, q \in X$ such that there exists one $\mathcal{P}-$transverse geodesic γ from p to q. Then γ is the only geodesic from p to q.*

Proof. Consider a geodesic γ' from p to q. If γ' is different from γ, then a simple loop obtained as the union of non-trivial subgeodesics of γ, γ' is easily found. This loop is contained in some $P \in \mathcal{P}$, so γ cannot be $\mathcal{P}-$transverse.

□

Chapter 5

Relatively hyperbolic groups

In this chapter we will present the "classical" definition of relatively hyperbolic group, work out a new definition and then use the latter to prove that negatively curved finite volume manifolds have relatively hyperbolic fundamental group. These fundamental groups are the motivating examples of relatively hyperbolic groups.

5.1 Asymptotically tree-graded spaces

Throughout the section, let X denote a metric space and let $\mathcal{P}$ be a collection of subsets of X. Roughly, X is asymptotically tree-graded if each asymptotic cone of X is tree-graded. However, we have to be more precise about the set of pieces in the asymptotic cones.

Definition 5.1. *X is asymptotically tree-graded with respect to $\mathcal{P}$ if each asymptotic cone Y of X is tree-graded with respect to the collection of the non-empty subsets of Y induced by elements of ${}^*\mathcal{P}$. Also, we require that, if 2 distinct elements of ${}^*\mathcal{P}$ induce pieces of Y, these pieces are distinct.*

Asymptotically tree-graded spaces were first defined in [DS], see Definition 4.19. Notice that the above definition is more easily stated, thanks to the nonstandard formalism.

We are finally ready to define relatively hyperbolic groups. Let G be a finitely generated group and let $H_1, \dots, H_n$ be subgroups of G.

Definition 5.2. *G is hyperbolic relative to $H_1, \dots, H_n$ if for some (and then every) finite generating system S for G, $C_S(G)$ is asymptotically tree-graded with respect to $\{gH_i | g \in G, i = 1, \dots, n\}$.*

The definition above is taken from [DS]. Several definitions of relative hyperbolicity appeared before this one, see for example [Fa] and [Os1]. Those definitions, however, are not fully "geometric" in that they are not based on the geometry of the Cayley graph, but on the geometry of other graphs obtained from the Cayley graph collapsing, in some way, the lateral classes of $H_1, \dots, H_n$. Another geometric definition can be found in [Dr2].

The remainder of this section is dedicated to the results regarding asymptotically tree-graded spaces that we will need. Let X be asymptotically tree-graded with respect to $\mathcal{P}$. Let us start with property (α_1) and (a slight modification of) property (α_2) as in Theorem 4.1 in [DS].

Lemma 5.3. *For each $H \geq 0$ there exists B such that diam$(N_H(P) \cap N_H(Q)) \leq B$ for each $P, Q \in \mathcal{P}$ with $P \neq Q$.*

Proof. It is an easy consequence of property (T_1) in the asymptotic cones of X (see Lemma 4.7 in [DS]). $\square$

Lemma 5.4. *For each $C \geq 0$ there exists M with the following property. If γ is a $(1, C)-$quasi-geodesic connecting x to y, and $d(x, P), d(y, P) \leq d(x, y)/3$ for some $P \in \mathcal{P}$, then $\gamma \cap N_M(P) \neq \emptyset$. Also, there exists σ such that, for each $C \geq 1$, M can be chosen to be σC.*

Proof. It is enough to prove the second part of the statement. Suppose that there is no such σ. Then we can find $C \geq 1$ (possibly infinite), an infinite σ, $x', y' \in {}^*X$, a $(1, C)-$quasi-geodesic γ' connecting them and $P \in {}^*\mathcal{P}$ such that $d(x', P), d(y', P) \leq d(x', y')/3$, but $\gamma' \cap N_{\sigma C}(P) = \emptyset$. Also, we can replace x', y' with a pair of points $x, y \in \gamma'$ such that $d(x, P), d(y, P) \leq d(x, y)/3$ and for each $x'', y'' \in \gamma'$ with $d(x'', y'') \leq d(x, y) - 1$, we have $d(x'', P) > d(x'', y'')/3$ or $d(y'', P) > d(x'', y'')/3$ (in fact, if γ is the $(1, C)-$quasi-geodesic from x to y obtained restricting γ', we clearly still have $\gamma' \cap N_{\sigma C}(P) = \emptyset$). We can also assume, up to modifying σ, that $d(\gamma, P) = \sigma C + 1$.

Suppose that $\nu = \max\{d(x, P), d(y, P)\} = d(x, P)$ and consider $Z = C(X, x, \nu)$ (ν is infinite as $d(\gamma, P) = \sigma C + 1$). Let Q be the piece induced by P in Z. Also, set $a = [x]$. Notice that $\sigma C + 1 = d(\gamma, P) \leq \nu$ and therefore $C \in o(\nu)$, as clearly $C \in o(\sigma C)$. Hence γ induces a geodesic δ in Z.

Let us show that $\delta \cap Q \neq \emptyset$. There are 2 cases to consider.

If $d(x, y) \in O(\nu)$, set $b = [y]$. By the fact that $d(a, Q), d(b, Q) \leq d(a, b)/3$, it is easily seen that $\pi_Q(a) \neq \pi_Q(b)$, and so we can apply Corollary 4.6.

If $d(x, y) \gg \nu$, δ is a geodesic ray. If the distance between $\delta(t)$ and Q is bounded for each t, then $\delta \cap Q \neq \emptyset$ by Corollary 4.5. Let us show that this must be the case. In fact, if $d(\delta(t), Q)$ is not bounded we can find $z \in \gamma$ such that $d(z, P) = \nu' \gg \nu$. Up to changing z, we can assume that $d(z, P) \geq \max\{d(w, P) | w \in \gamma\} - 1$. Consider $Z' = C(X, z, \nu')$ and let Q' be the piece induced by P in Z'. As $C \in o(\nu')$, we have that γ may

- induce a geodesic δ in Z'. The endpoints of δ lie on Q', but δ contains a point outside Q'. This is in contradiction with the convexity of Q'.

- induce a geodesic ray δ in Z', which stays at bounded distance from Q' and contains a point outside Q'. But Corollary 4.5, the fact that the starting point of δ lies on Q' and the convexity of Q' imply that δ cannot contain a point outside Q', a contradiction.
- induce a geodesic line δ in Z', which stays at bounded distance from Q' and contains the point $[z]$ outside Q'. We have that both geodesic rays contained in δ and starting from $[z]$ intersect Q'. This, together with the convexity of Q', is in contradiction with $[z] \notin Q'$.

Consider a point $w \in \gamma$ such that $[w] = \pi_Q(a)$. As we have $st(d(x,w)/\nu) = 1$ and $d(x,y) \geq d(x,w) + d(w,y) - 3C$, we also have $d(w,z) < d(x,y) - 1$ (recall that $C \in o(\nu)$).

Now, if we have $d(w,y) \gg \nu$, it is clear that $d(w,P), d(y,P) \leq d(w,y)/3$, for the left-hand sides are in $o(\nu)$, while $d(w,z) \gg \nu$. This contradicts our choice of x, y.

Consider the other case, that is $d(w,y) \equiv \nu$ (it cannot happen that $d(w,y) \in o(\nu)$, for otherwise we would have $st(d(x,P)/\nu) = st(d(x,y)/\nu)$, in contradiction with $d(x,P) \leq d(x,y)/3$). In this case consider $v \in \gamma$ such that $[v] = \pi_Q(y)$. Notice that $d(w,v) \equiv \nu$ (as $st(d(x,y)/\nu) = st(d(x,w)/\nu) + st(d(w,v)/\nu) + st(d(v,y)/\nu)$ and $st(d(x,w)/\nu), st(d(v,y)\nu) \leq st(d(x,y)/(3\nu))$). Therefore, we have $d(w,P), d(v,P) \leq d(w,v)/3$, for $d(w,P), d(v,P) \in o(\nu)$, once again in contradiction with our choice of x, y (as, by an easy argument, $d(w,v) < d(x,y) - 1$).

□

Remark 5.5. The most frequent use of the previous lemma will be that if the *geodesic $\hat{\delta} \subseteq {}^*X$ induce a geodesic δ in an asymptotic cone of X which intersects the piece induced by $P \in {}^*\mathcal{P}$ in a non-trivial subgeodesic, then $\hat{\gamma} \cap N_M(P) \neq \emptyset$. In fact, consider the sub-*geodesic γ of $\hat{\delta}$ with endpoints x, y such that $[x], [y] \in Q$ and $[x] \neq [y]$, where Q is the piece induced by P. Then $d(x,y) \equiv \nu$, and $d(x,P), d(y,P) \ll \nu$, so $d(x,P), d(y,P) \leq d(x,y)$ and we can apply the lemma.

We will also need that each $P \in \mathcal{P}$ is *quasi-convex*, in the following sense (see Lemma 4.3 in [DS]):

Lemma 5.6. *There exists t such that for each $L \geq 1$ each geodesic connecting $x, y \in N_L(P)$ is contained in $N_{tL}(P)$.*

Proof. If the thesis were false we could find an asymptotic cone of X and either a geodesic connecting 2 points in a piece not entirely contained in a piece, or a geodesic ray at bounded distance from a certain piece, but not intersecting it.

□

We will need some consequences of Lemma 4.9. The statements below tell us, very roughly, that if two geodesics start diverging, then they will continue to diverge.

Let us fix the notation for the following lemmas. Consider *geodesics in $^*X$ $\hat{\alpha}$ and $\hat{\beta}$ from p to q and from p to r, respectively. Suppose that there is a scaling factor ν such that in $Y = C(X, p, \nu)$ the geodesic (rays) α^{-1}, β induced by $\hat{\alpha}^{-1}$, $\hat{\beta}$ concatenate well.

Lemma 5.7. *There exists an infinitesimal η such that for each $t_1, t_2 \geq \nu$ (t_1 in the domain of $\hat{\alpha}$, t_2 in the domain of $\hat{\beta}$), $d(\hat{\alpha}(t_1), \hat{\beta}(t_2)) \geq (1-\eta)(t_1+t_2)$.*

Proof. Suppose that this is not the case. Then there exists $\epsilon \in \mathbb{R}^+$ and $t_1, t_2 \geq \nu$ such that $d(\hat{\alpha}(t_1), \hat{\beta}(t_2)) \leq (1-\epsilon)(t_1+t_2)$ (notice that for all such pairs $t_1 \equiv t_2$). We can choose a pair t_1, t_2 satisfying that property with $\tau = \max\{t_1, t_2\}$ minimal. By Lemma 4.9, $\tau \notin O(\nu)$, because the concatenation $\alpha^{-1}\beta$ is a geodesic. Set $Z = C(X, p, \tau)$ and $p_1 = [\hat{\alpha}(t_1)]$, $p_2 = [\hat{\beta}(t_2)]$. By the minimality of τ, the geodesic triangle with sides $[p_1, p_2]$ and the geodesics α', β' induced by $\alpha|_{[0,t_1]}$, $\beta|_{[0,t_2]}$ is not a tripod. In fact, if it was a tripod, we would have $[\hat{\alpha}(t_1')] = [\hat{\beta}(t_2')] \neq [p]$ for some $t_i' < t_i$ (notice that it cannot be a tripod by contained in a geodesic from p_1 to p_2 as $d(\hat{\alpha}(t_1), \hat{\beta}(t_2)) \leq (1-\epsilon)(t_1+t_2)$), so $d(\hat{\alpha}(t_1'), \hat{\beta}(t_2')) \in o(t_1'+t_2')$. For the same reason, $\alpha' \cap \beta' = \{p\}$. Therefore, there exists a piece, induced by, say, $P \in \mathcal{P}$ which contains initial subsegments of α', β'. Let M be as in Lemma 5.4. We have that $\hat{\alpha}$ and $\hat{\beta}$ intersect $N_M(P)$ (see Remark 5.5). Let $\hat{\alpha}(s_1), \hat{\beta}(s_2)$ be the first points in $\hat{\alpha} \cap N_M(P)$ and $\hat{\beta} \cap N_M(P)$ and set $\tau' = \max\{s_1, s_2\}$ (suppose $\tau' = s_1$). Notice that $\tau' \in o(\tau)$ (see once again Remark 5.5). Set $W = C(X, p, \tau')$ and let α'', β'' be the geodesic rays induced by $\hat{\alpha}, \hat{\beta}$ in W. Let Q be the piece induced by P in W. Notice that $\pi_Q([p]) = [\hat{\alpha}(s_1)]$, for otherwise there would be a non-trivial subpath of α'' contained in Q, contradicting the minimality of s_1 (once again, by Remark 5.5). Similarly, $\pi_Q([p]) = [\hat{\beta}(s_2)]$. Notice that we just proved the following remark.

Remark 5.8. Suppose that $\hat{\gamma}$ is a * geodesic starting from p. Also, suppose that in an asymptotic cone with basepoint p, $\hat{\gamma}$ induces a geodesic γ intersecting the piece Q induced by $P \in \mathcal{P}$ in a non-trivial subsegment. Then $\pi_Q([p]) = \gamma(t)$, where t is minimal such that $\gamma(t) \in N_M(P)$.

This, together with $d([p], [\hat{\alpha}(s_1)]) = 1$, implies that $d(\hat{\alpha}(s_1), \hat{\beta}(s_2)) \in o(s_1+s_2)$, contradicting the minimality of τ (as $\tau' \ll \tau$), *unless* $s_1 < \nu$ or $s_2 < \nu$. However, this is not the case, as we are going to show. First, notice that $[\hat{\alpha}(s_1)] = [\hat{\beta}(s_2)] \neq [p]$ implies $s_1 \equiv s_2$. Therefore, if $s_1 < \nu$ or $s_2 < \nu$, they are both in $O(\nu)$, and if one of them is in $o(\nu)$, they are both in $o(\nu)$.

Let us consider 2 cases.

If $s_i \equiv \nu$, in Y we would have $[p] \neq \pi_{Q'}([p]) \in \alpha \cap \beta$, where Q' is the piece induced by P, contradicting condition (2) in the definition of concatenating well.

If $s_i \in o(\nu)$, there is a contradiction with condition (3) in the definition of concatenating well, as we would have that initial subpaths of α, β would be contained in Q' (actually, they would be entirely contained in Q'). This follows from the quasi-convexity of P and the remark above, applied to both endpoints of long enough subgeodesics of $\hat{\alpha}, \hat{\beta}$ which induce in Z geodesics contained in the piece induced by P.

□

Lemma 5.9. *Consider $\mu \geq \nu$ and set $Z = C(X, p, \mu)$. If α', β' are the geodesic (rays) induced by $\hat{\alpha}, \hat{\beta}$ in Z, then α'^{-1} and β' concatenate well.*

Proof. From the previous lemma, we know that $\alpha'^{-1}\beta'$ is a geodesic. Therefore condition (2) in the definition of concatenating well (see Lemma 4.9) is guaranteed. If there is P which induce a piece containing $[p]$ and subgeodesics of α', β', consider s_1 and s_2 such that $\hat{\alpha}(s_1), \hat{\beta}(s_2)$ are the first points in $\hat{\alpha} \cap N_M(P), \hat{\beta} \cap N_M(P)$. Proceeding as in the previous lemma we find that $d(\hat{\alpha}(s_1), \hat{\beta}(s_2)) \in o(s_1 + s_2)$, so $s_1 < \nu$ or $s_2 < \nu$ by the previous lemma, and in this case the final part of the argument there applies verbatim.

□

Finally, the result we were actually aiming for.

Lemma 5.10. *$d(q, r) = d(q, p) + d(p, r) - \xi\nu$, for some infinitesimal ξ. Also, each geodesic $\hat{\gamma}$ from q to r induce in Y a geodesic (ray, line) containing $[p]$.*

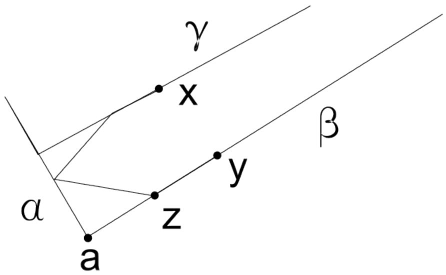
Proof. Notice that it is enough to prove the last part of the statement.

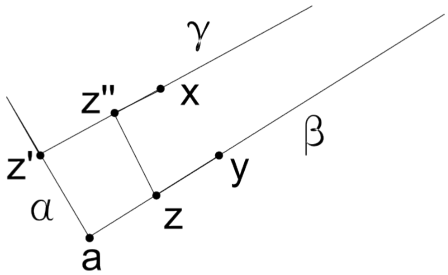
Now, we wish to prove that it is enough to consider the case that $d(p, q) \in O(\nu)$ or $d(q, r) \in O(\nu)$. In fact, consider $\hat{\gamma}$ as in the statement and set $\mu = d(\hat{\gamma}, p) = d(y, p)$, for some $y \in \hat{\gamma}$. Also, let $\hat{\delta}$ be a *geodesic from p to y. Set $Z = C(X, p, \mu)$ and let γ, δ be the geodesic induced by $\hat{\gamma}, \hat{\delta}$ in Z. Also, let α', β' be the geodesics induced by $\hat{\alpha}, \hat{\beta}$. By Remark 4.12 and Lemma 5.9, δ^{-1} and α' or δ^{-1} and β' concatenate well. In any case, if we knew the special case of the lemma (substituting ν with μ and $\hat{\alpha}$ or $\hat{\beta}$ with $\hat{\delta}$), we could conclude that γ contains $[p]$, contradicting $\mu = d(\hat{\gamma}, p)$.

We are left to prove the special case. Assume, without loss of generality, $d(p, q) \in O(\nu)$. If also $d(p, r) \in O(d)$, the lemma, granted Lemma 4.9, is trivial. Therefore assume that $\nu \in o(d(p, r))$.

Consider a *geodesic $\hat{\gamma}$ from q to r, and let γ be the geodesic ray in Y induced by it. We want to prove that there is a point on $\gamma \cap \alpha$, which implies the thesis, by the last part of the statement of Lemma 4.9. Suppose that this is not the case.

Let x be a point on γ and y a point on β. We want to prove that the concatenation of a geodesic from x to $a = [p]$ and the subpath of β from a to y is a geodesic. First, $[x,a] \cap \beta = \{a\}$. In fact, if this is false consider the point $z \in [x,a] \cap \beta$ closest to x. If $[x,z] \cap \alpha \neq \emptyset$, a simple geodesic triangle containing initial subpaths of β and α is easily constructed. This is in contradiction with condition (3) in Lemma 4.9, because such a triangle is contained in a piece. On the other hand, if $[x,z] \cap \alpha = \emptyset$, let z' be the point in $\alpha \cap \gamma$ closest to a and z'' the point in $\gamma \cap [x,z]$ closest to z. The quadrangle $[z',a],[a,z],[z,z''],[z'',z']$ is simple, and we get a contradiction as before.





Suppose that there is a piece P containing a final non-trivial subpath $[x',a]$ of $[x,a]$ and an initial non-trivial subpath of β. We have that $\pi_P(x) = x' \neq a$. But π_P is constant along the concatenation of α and γ, as this path intersects P only in its starting point (as α^{-1} and β satisfy condition (3) in the definition of concatenating well and we assumed $\gamma \cap \beta = \emptyset$). In particular $\pi_P(x) = \pi_P(a) = a$, a contradiction.

We proved that the 3 conditions of Lemma 4.9 are satisfied, therefore the concatenation of $[x,a]$ and $[a,y]$ is a geodesic. This implies that for each $x \in \gamma$, $y \in \beta$, $d(x,y) \geq d(x,b) + d(y,a) - d(a,b)$, where $b = [q]$. In particular, for each $t \in {}^*\mathbb{R}^+$, $t \geq 2d(p,q)$ and $t \equiv d$, $d(\hat{\gamma}(t), \hat{\beta}(t)) > t$ (as $st(d(\hat{\gamma}(t), \hat{\beta}(t))/\nu) \geq st(2t/\nu - d(p,q)/\nu) > st(t/\nu)$ because $st(d(p,q)/\nu) = d(a,b)$). On the other hand, the final point of $\hat{\gamma}$ is on $\hat{\beta}$, their length is greater than $2d(p,q)$ and the distance between their starting points is $d(p,q)$, therefore

$$\tau = \min\{t \geq 2d(p,q) : d(\hat{\beta}(t), \hat{\alpha}(t)) \leq t\}$$

exists, and $\tau \gg d$. Consider the asymptotic cone with scaling factor τ and basepoint p. Let γ' be the projection of $\hat{\gamma}|_{[0,\tau]}$ and β' be the projection of $\hat{\beta}|_{[0,\tau]}$. Consider a geodesic triangle which contains β' and α'. By minimality of τ, it cannot be a tripod and, also, initial subpaths of β' and α' are contained in a piece P, induced by, say, $Q \in \mathcal{P}$.

Let M be as in Lemma 5.4. We have that $\hat{\beta}$ and $\hat{\gamma}$ intersect $N_M(P)$. Let r (resp. s) be the first point in $\hat{\beta} \cap N_M(P)$ (resp. $\hat{\gamma} \cap N_M(P)$). Set $\delta = \max\{d(q,r), d(q,s)\}$ (notice that $\delta \in o(t_1)$). It is impossible that $\delta \in o(\nu)$, for otherwise γ would pass through a. Also $\delta \equiv \nu$ cannot hold, for otherwise

β and $\alpha\gamma$ would intersect in the projection of $[p]$ on the piece induced by Q. We are left to show that the last case, $\delta \gg \nu$, cannot hold as well. Consider the asymptotic cone Z of G with basepoint p and scaling factor δ. If β'', γ'' are the geodesic rays induced by $\hat{\beta}, \hat{\gamma}$ and P' is the piece induced by Q in Z, we have that $\pi_{P'}([p]) \in \beta'' \cap \gamma''$. This is easily seen to contradict the minimality of t.

□

Corollary 5.11. *Consider * geodesics $\hat{\alpha}$ and $\hat{\beta}$ connecting, respectively, p to p' and q to q', where $d(p,q) \gg 1$. Let $\hat{\delta}$ be a *geodesic from p to q. Let α, β, δ the geodesics induced in $Y = C(X, p, d(p,q))$. Suppose that δ^{-1}, α and δ, β concatenate well. Then $d(p', q') = d(p', p) + d(p,q) + d(q, q') - \rho$, for some $\rho \in o(d(p,q))$. Also, any *geodesic from p' to q' induces a geodesic in Y containing $[p], [q]$.*

Proof. We just need to apply the previous lemma twice.

□

5.2 Alternative definition

In this section we state the analogue of the alternative definition of tree-graded spaces we gave before. Throughout the section let X be a geodesic metric space and let $\mathcal{P}$ be a collection of quasi-convex subsets of X such that $\bigcup_{P \in \mathcal{P}} P$ is $k-$dense in X for some $k \geq 0$.

We will need the coarse versions of the definitions of projection system and being transverse-free.

Definition 5.12. A family of maps $\Pi = \{\pi_P : X \to P\}_{P \in \mathcal{P}}$ will be called almost-projection system for $\mathcal{P}$ if there exist $C \geq 0$ such that, for each $P \in \mathcal{P}$:,

$(AP1)$ for each $r \in P$, $z \in X$, $d(r,z) \geq d(r, \pi_P(z)) + d(\pi_P(z), z) - C$,
$(AP2)$ for each $z \in X$ with $d(z,P) = d$, $diam\,(\pi_P(B_d(x))) \leq C$,
$(AP3)$ for each $P \neq Q \in \mathcal{P}$, $diam(\pi_P(Q)) \leq C$.

Remark 5.13. For each $x \in X$ and $P \in \mathcal{P}$, $d(x, \pi_P(x)) \leq d(x,P) + C$.

Now, let us prove coarse versions of Lemma 4.15$-$(1) and Lemma 4.16.

Lemma 5.14. *Suppose that $\{\pi_P\}$ is an almost projection system. For each M there exists μ such that each geodesic γ from $x \in X$ to $y \in N_M(P)$, for some $P \in \mathcal{P}$, intersects $B_\mu(\pi_P(x))$.*

Proof. Let y' be the last point in $\gamma \cap \overline{B}_d(x)$, where $d = d(x,P)$, and let γ' be the subgeodesic of γ from x to y'. As $d(y, \pi_P(y)) \leq M + C$ and $d(\pi_P(y'), \pi_P(x)) \leq C$, we have that

$d(y', y) \geq d(y', \pi_P(y)) - M - C \geq d(y', \pi_P(y')) + d(\pi_P(y'), \pi_P(y)) - M - 2C \geq$

$$d(y', \pi_P(x)) + d(\pi_P(x), \pi_P(y)) - M - 4C.$$

Also,

$$d(x, y) \leq d(x, \pi_P(x)) + d(\pi_P(x), \pi_P(y)) + M + C.$$

As $d(x, y) = d(x, y') + d(y', y)$ and $d(x, y') = d(x, P)$, we obtain

$$d(y', \pi_P(x)) + d(\pi_P(x), \pi_P(y)) - M - 4C + d(x, P) \leq$$

$$d(x, \pi_P(x)) + d(\pi_P(x), \pi_P(y)) + M + C \leq d(x, P) + d(\pi_P(x), \pi_P(y)) + M + 2C.$$

Therefore,

$$d(y', \pi_P(x)) \leq 2M + 6C.$$

We are done setting $\mu = 2M + 6C$.

□

We will consider the following coarse analogs of properties $(P'1)$ and $(P'2)$.

$(AP'1)$ There exists $C \geq 0$ such that for each $z \in X$, $d(z, \pi_P(z)) \leq d(z, P) + C$.

$(AP'2)$ There exists $C \geq 0$ with the property that for each $z_1, z_2 \in X$ such that $d(\pi_P(z_1), \pi_P(z_2)) \geq C$, we have

$$d(z_1, z_2) \geq d(z_1, \pi_P(z_1)) + d(\pi_P(z_1), \pi_P(z_2)) + d(\pi_P(z_2), z_2) - C.$$

Lemma 5.15. *$(AP1) + (AP2) \iff (AP'1) + (AP'2)$.*

Proof. $\Leftarrow$: Fix C large enough so that $(AP'1), (AP'2)$ hold. Property $(AP1)$ is not trivial only if $d(\pi_P(z), z)$ is large, and in this case it follows from $(AP'2)$ setting $z_1 = z$ and $z_2 = r$ and keeping into account $d(\pi_P(r), r) \leq C$. Let us show property $(AP2)$. Notice that $d(\pi_P(z), \pi_P(z')) > C$ implies $d(z, z') > d(z, P) - 2C$. We want to exploit this fact. Set $d = d(z, P)$. Notice that if $z' \in B(z, d)$, then there exists $z'' \in B_{d-2C}$ such that $d(z', z'') \leq 2C$ and one of of the following 2 cases holds:

- $z' \in N_{6C}(P)$, or
- $d(z'', P) \geq 4C$.

In the first case either $d(\pi_P(z'), \pi_P(z'')) < C$ or

$$d(z', \pi_P(z')) + d(\pi_P(z'), \pi_P(z'')) + d(\pi_P(z''), z'') - C \leq d(z', z'') \leq 2C,$$

and so $d(\pi_P(z'), \pi_P(z'')) \leq 3C$. In the second case $d(z', z'') \leq d(z', P) - 2C$, and so $d(\pi_P(z'), \pi_P(z'')) \leq C$.

These considerations yield $diam\,(\pi_P(B_d(x)))) \leq 4C$.

$\Rightarrow$: We already remarked that $(AP'1)$ holds. Let $C > 0$ be large enough so that $(AP1), (AP2)$ hold. Consider any geodesic γ with endpoints x, y

such that $\gamma \cap N_{2C}(P) = \emptyset$, for some $P \in \mathcal{P}$. We want to prove that $d(\pi_P(x), \pi_P(y)) \leq d(x,y)/2 + C$. This is easily done considering a partition of γ in subgeodesics $\gamma_i = [x_i, y_i]$ of length $2C$ and one subgeodesic $\gamma' = [x', y']$ of length at most $2C$. In fact, by property $(AP2)$ we have $d(\pi_P(x_i), \pi_P(y_i)) \leq C = d(x_i, y_i)/2$ and $d(\pi_P(x'), \pi_P(y')) \leq C$, so

$$d(\pi_P(x), \pi_P(y)) \leq \sum d(\pi_P(x_i), \pi_P(y_i)) + d(\pi_P(x'), \pi_P(y')) \leq$$

$$\sum d(x_i, y_i)/2 + d(x', y')/2 + C = d(x,y)/2 + C.$$

Consider points z_1, z_2 such that $d(\pi_P(z_1), \pi_P(z_2)) \geq 6C+1$, for some $P \in \mathcal{P}$, and let γ be a geodesic connecting them. We want to show that $\gamma \cap N_{2C}(P) \neq \emptyset$. If we do so, property $(AP'2)$ follows from Lemma 5.14, which guarantees that there exists μ such that γ intersects $B_\mu(\pi_P(z_i))$.

Set $d_i = d(z_i, P)$. We have that $B_{d_1}(z_1) \cap B_{d_2}(z_2) \cap \gamma = \emptyset$, for otherwise we would have $d(\pi_P(z_1), \pi_P(z_2)) \leq 2C$. Let x_i be the point on γ whose distance from z_i is d_i. Suppose by contradiction that $[x_1, x_2] \cap N_{2C}(P) = \emptyset$. Then $d(\pi_P(x_1), \pi_P(x_2)) \leq d(x_1, x_2)/2 + C$ and in particular $d(x_1, x_2)/2 \geq 5C + 1$. So,

$$d(z_1, z_2) \leq d(z_1, \pi_P(z_1)) + d(\pi_P(z_1), \pi_P(x_1)) + d(\pi_P(x_1), \pi_P(x_2)) +$$

$$d(\pi_P(x_2), \pi_P(z_2)) + d(\pi_P(z_2), z_2) \leq$$

$$(d(z_1, P) + C) + C + (d(x_1, x_2)/2 + C) + C + (d(z_2, P) + C) \leq$$

$$d(z_1, x_1) + d(x_1, x_2) + d(x_2, z_2) + 5C - d(x_1, x_2)/2 < d(z_1, z_2),$$

a contradiction. Therefore $[x_1, x_2] \cap N_{2C}(P) \neq \emptyset$ and in particular $\gamma \cap N_{2C}(P) \neq \emptyset$, as required.

□

Definition 5.16. A $(1, C)-$quasi-geodesic triangle Δ is $\mathcal{P}-$almost-transverse with constants K, D if, for each $P \in \mathcal{P}$ and each side γ of Δ, $diam(N_K(P) \cap \gamma) \leq D$.

$\mathcal{P}$ is asymptotically transverse-free if there exist λ, σ such that for each $C, D \geq 1$, $K \geq \sigma C$ the following holds. If Δ is a $(1, C)-$quasi-geodesic triangle which is $\mathcal{P}-$almost-transverse with constants K, D, then Δ is $\lambda(D + C)-$thin.

Theorem 5.17. *X is asymptotically tree-graded with respect to $\mathcal{P}$ if and only if $\mathcal{P}$ is asymptotically transverse-free and there exists an almost projection system for $\mathcal{P}$.*

Proof. $\Leftarrow$: Consider an asymptotic cone $Y = C(X, p, \nu)$ of X and consider the collection $\mathcal{P}'$ of the sets induced by elements of ${}^*\mathcal{P}$ in Y. It is quite clear that elements of $\mathcal{P}'$ cover Y and that they are geodesic, by the assumptions on $\mathcal{P}$. Also, it is very easy to see that an almost projection system for $\mathcal{P}$ induces a projection system for $\mathcal{P}'$.

Let us prove that $\mathcal{P}'$ is transverse-free. Consider a geodesic triangle Δ in Y. We would like to say that it is induced by a *geodesic triangle in $^*X$. This is not the case, but, as shown in the following lemma, it is not too far from being true.

Lemma 5.18. *Any geodesic* $\gamma : [0, l] \to Y$ *is induced by an internal* $(1, \rho\nu)-$*quasi-geodesic in* $^*X$*, where* $\rho \ll 1$.

Proof. For each $q \in S = \{l\} \cup (\mathbb{Q} \cap [0, l])$ choose some $x_q \in {}^*X$ which projects on $\gamma(q)$. We can choose an infinitesimal ξ such that $(|q_2 - q_1| - \xi)\nu \leq d(x_{q_1}, x_{q_2}) \leq (|q_2 - q_1| + \xi)\nu$ for each $q_1, q_2 \in S$. Let $Q \subseteq S$ be a finite set. We want to show that there exists an infinitesimal ρ_Q and an internal $(1, \rho_Q\nu)-$quasi-geodesic $\delta_Q : [0, l\nu] \to {}^*X$ which contains each x_q for $q \in Q$. Set $Q = \{q_0, \dots, q_n\}$, where $q_i < q_j \iff i < j$. Suppose, for convenience, $q_0 = 0$ and $q_n = l$. Let δ_Q be the concatenation of *geodesics (suitably reparametrized) $\delta_k : [q_k\nu, q_{k+1}\nu] \to {}^*X$. We have, for $x \in [q_i\nu, q_{i+1}\nu]$, $y \in [q_j\nu, q_{j+1}\nu]$, for some $i < j$,

$$d(\delta(x), \delta(y)) \leq (q_{i+1}\nu - x) + d(x_{q_{i+1}}, x_{q_j}) + (y - q_j\nu) \leq$$

$$(q_{i+1}\nu - x) + (q_j - q_{i+1} + \xi)\nu + (y - q_j\nu) = (y - x) + \xi\nu.$$

Also, clearly $l(\delta|_{[q_i\nu, q_{j+1}\nu]}) \leq (q_{j+1} - q_i)\nu + (j + 1 - i)\xi\nu \leq (q_{j+1} - q_i)\nu + n\xi\nu$. Therefore,

$$l(\delta|_{[q_i\nu,x]}) + l(\delta|_{[x,y]}) + l(\delta|_{[y,q_{j+1}\nu]}) = l(\delta|_{[q_i\nu,q_{j+1}\nu]}) \leq d(x_{q_i}, x_{q_{j+1}}) + (n+1)\xi\nu \leq$$

$$d(x_{q_i}, \delta(x)) + d(\delta(x), \delta(y)) + d(\delta(y), x_{q_{j+1}}) + (n + 1)\xi\nu.$$

As $l(\delta|_{[q_i,x]}) \geq d(x_{q_i}, \delta(x))$ and $l(\delta|_{[y,x_{q_{j+1}}]}) \geq d(\delta(y), x_{q_{j+1}})$, we conclude that

$$d(\delta(x), \delta(y)) \geq l(\delta|_{[x,y]}) - n\xi\nu.$$

Finally, $l(\delta|_{[x,y]}) \geq (q_{i+1}\nu - x) + (q_j - q_{i+1})\nu + (y - q_j\nu) = y - x$. Therefore $d(\delta(x), \delta(y)) \geq (y - x) - (n + 1)\xi\nu$.

The case $j < i$ is analogous and the case $i = j$ is even easier to handle, so we have that δ is an internal $(1, \rho_Q\nu)-$quasi-geodesic for $\rho_Q = (n + 1)\xi$. Using $\aleph_0-$saturation we get that for any infinitesimal ρ such that $\rho \geq \rho_Q$ for each Q as above, the set of internal $(1, \rho\nu)-$quasi-geodesics from x_0 to x_l containing x_q for each $q \in S$ is non-empty. Such an internal quasi-geodesics clearly induce γ. $\square$

Using this lemma, we obtain that Δ, the geodesic triangle we are considering, is induced by some $(1, \rho\nu)-$quasi-geodesic triangle $\hat{\Delta}$ in $^*X$. If Δ is $\mathcal{P}'-$transverse, for each $K \in o(\nu)$ there exists $D \in o(\nu)$ such that $\hat{\Delta}$

is $\mathcal{P}-$almost-transverse with constants K, D. In particular we can choose $K \geq \sigma C$, but $K \in o(\nu)$, and so we have that $\hat{\Delta}$ is $\kappa-$thin, where $\kappa = \lambda(D+C) \in o(\nu)$. This implies that Δ is a tripod. This proves that $\mathcal{P}'$ is transverse-free. We proved that both conditions of Theorem 4.18 are satisfied for Y and $\mathcal{P}'$, therefore Y is tree-graded with respect to $\mathcal{P}'$. As Y was any asymptotic cone of X, the proof is complete.

$\Rightarrow$: For each $P \in \mathcal{P}$, define π_P in such a way that for each $x \in X$ we have $d(\pi_P(x), x) \leq d(x, P) + 1$. This definition is just slightly different from Definition 4.9 in [DS]. Property $(AP1)$ is obvious.

Lemma 5.19. *There exists R such that each geodesic γ from x to $N_M(P)$ intersects $\overline{B}_R(\pi_P(x))$, for each $x \in X$ and $P \in \mathcal{P}$.*

Proof. We shall show that there is a bound on $d(y, \pi_P(x))$, where y is the first point in $\gamma \cap P$ for γ as in the statement. Suppose that there does not exist such a bound. Then there exists an infinite ν, a *geodesic from $x \in {}^*X$ to some point p in $\hat{P} \in {}^*\mathcal{P}$ such that $d(y, \pi_{\hat{P}}(x)) = \nu$, for $y \in \gamma$ as above. Consider $Y = C(X, \pi_{\hat{P}}(x), \nu)$ and let P be the piece induced by $\hat{P}$. Clearly, $[y] \in P$. By the property of M, it is easily shown that $[x, y]$ induces a geodesic (or geodesic ray) which intersects P only in $[y]$. Therefore, by Lemma 5.10, $d(\pi_{\hat{P}}(x), x) = d(\pi_{\hat{P}}(x), y) + d(y, x) - \epsilon$, for $\epsilon \in o(\nu)$. But this implies $d(x, \hat{P}) \leq d(x, y) + M \leq d(x, \pi_{\hat{P}}(x)) - \nu/2 + M$, in contradiction with the defining property of $\pi_{\hat{P}}$.

□

The following lemma clearly implies $(AP2)$.

Lemma 5.20. *There exists L such that for each $x, y \in X$, $P \in \mathcal{P}$, if $d(\pi_P(x), \pi_P(y)) \geq L$, then any geodesic from x to y intersects $B_L(\pi_P(y))$ and $B_L(\pi_P(y))$.*

Proof. Suppose that the statement is false. Then there exists some infinite ν, $x, y \in {}^*X$, a *geodesic $\hat{\gamma}$ connecting them and $P \in {}^*\mathcal{P}$ such that $d(\pi_P(x), \pi_P(y)) = \nu$, and $d(\hat{\gamma}, \pi_P(x)) \geq \nu$ or $d(\hat{\gamma}, \pi_P(y)) \geq \nu$. Consider $Y = C(X, \pi_P(x), \nu)$. Let $\hat{\alpha}$ be a *geodesic from x to $\pi_P(x)$ and let α be the induced geodesic in Y. Define similarly $\hat{\beta}$ from y to $\pi_P(y)$ and β. We have that α and β intersect the piece Q induced by P only in one endpoint. In fact, if, say, $\alpha \cap Q$ was a non-trivial subpath, we could find a point on $\hat{\alpha}$ with infinite distance from $\pi_P(x)$, but in $\hat{\alpha} \cap N_M(P)$. An initial subgeodesic of $\hat{\alpha}$ would therefore contradict Lemma 5.19. So, it is easily seen that α, β and δ, a geodesic in Y induced by a *geodesic from $\pi_P(x)$ to $\pi_P(y)$, satisfy the hypotheses of Corollary 5.11. Applying that corollary, we have $d(\hat{\gamma}, \pi_P(x)), d(\hat{\gamma}, \pi_P(y)) \in o(\nu)$, in contradiction with $d(\hat{\gamma}, \pi_P(x)) \geq \nu$ or $d(\hat{\gamma}, \pi_P(y)) \geq \nu$.

□

Let us prove $(AP3)$ (we will use the lemma once again). Let B be a uniform bound on the diameters of $N_H(P) \cap N_H(Q)$ for $P \neq Q \in \mathcal{P}$ (see Lemma 5.3), where $H = \max\{tM, L\}$ for t as in Lemma 5.6. Fix $P, Q \in \mathcal{P}$, $P \neq Q$. Suppose that there exist $x, y \in Q$ such that $d(\pi_P(x), \pi_P(y)) \geq 2L + B + 1$. Consider a geodesic $[x, y]$. It is contained in $N_{tM}(Q)$. Consider points x', y' on $[x, y]$ such that $d(x', \pi_P(x)) \leq L$, $d(y', \pi_P(y)) \leq L$. Then $d(x', y') \geq d(\pi_P(x), \pi_P(y)) - 2L \geq B + 1$. This is in contradiction with $diam(N_H(P) \cap N_H(Q)) \leq B$.

These considerations readily imply $(AP3)$.

We are left to show that $\mathcal{P}$ is asymptotically transverse-free. We will use Lemma 5.4. Consider σ as in that lemma. Suppose that there is no λ such that $\mathcal{P}$ satisfies the definition of being asymptotically transverse-free with the given σ. Then we can find an infinite ν, a $(1, C)-$quasi-geodesic triangle which is $^*\mathcal{P}-$almost-transverse with constants K, D ($C, D \geq 1$, $K \geq \sigma C$, possibly infinite) such that its optimal thinness constant is $\tau = \nu(D + C)$. Therefore, if γ_i, $i = 0, 1, 2$, are the sides of Δ, we have $\gamma_i \subseteq N_\tau(\gamma_{i-1} \cup \gamma_{i+1})$ and there exists y in, say, γ_0 such that $d(y, \gamma_1), d(y, \gamma_2) \geq \tau - 1$. Consider $Y = C(X, y, \tau)$. We want to show that each γ_i induces a geodesic δ_i in Y contained in a transversal tree (they indeed induce geodesics as $C \in o(\tau)$). In fact, suppose that the piece induced by $P \in \mathcal{P}$ intersects δ_i in a non-trivial subgeodesic δ. We have that on γ_i between each $p, q \in \gamma_i$ such that $[p], [q] \in \delta$ and $[p] \neq [q]$ there exists a point $x \in N_{\sigma C}(P)$. This implies that $d = diam(\gamma_i \cap N_{\sigma C}(P)) \equiv \tau$, and so $d \gg D$, in contradiction with our $^*\mathcal{P}-$almost-transversality assumption on Δ.

The proof is complete.

□

We wish to substitute property $(AP2)$ with a weaker property, which will be easier to prove. Define the following property (see Lemma 4.11 in [DS]):

(AP_w2) there exists $f : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ with $\lim_{x \to +\infty} f(x)/x = 0$ such that, for each $x \in X$ and $P \in \mathcal{P}$, $diam(\pi_P(B_{d/2}(x)) \leq f(d)$, where $d = d(x, P)$.

The following proposition follows easily from the fact that (AP_w2) is clearly weaker than $(AP2)$ and from the proof of the theorem above.

Proposition 5.21. *X is asymptotically tree-graded with respect to $\mathcal{P}$ if and only if $\mathcal{P}$ is asymptotically transverse-free and there exists a family of maps $\{\pi_P\}_{P \in \mathcal{P}}$ satisfying $(AP1)$, (AP_w2) and $(AP3)$.*

5.3 Finite volume manifolds of negative curvature

The aim of this section is to provide examples of finite volume negatively curved manifolds and to suggest their importance.

Throughout this section all Riemannian manifolds are implied to be connected, orientable and complete. By negatively curved manifold we mean a manifold whose sectional curvatures at each point are all negative and bounded from below and from above by $-b^2$ and $-a^2$, for some $a, b > 0$. The examples of negatively curved manifolds which we will be mostly interested in are hyperbolic manifolds. Let M denote, until the end of the chapter, a finite volume manifold of negative curvature. Examples of finite volume hyperbolic manifolds are provided by the following theorem (we will provide more interesting examples later).

Theorem 5.22. *A surface of finite type (a compact surface with a finite number of points removed) admits a hyperbolic metric of finite volume if and only if its Euler characteristic is negative.*

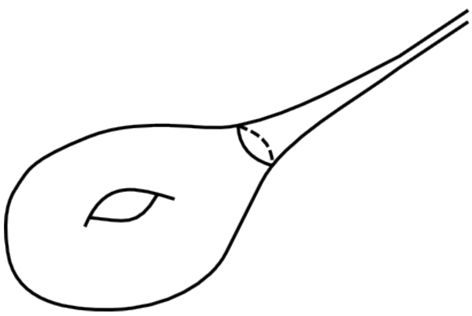


Figure 5.1: A finite volume hyperbolic surface

Let us study the structure of M. Let us first do it in the case that M is hyperbolic, following [BP].

Definition 5.23. Fix $\epsilon > 0$. Set

$$M_{(0,\epsilon]} = \{x \in M | \exists [\sigma] \in \pi_1(M, x) \backslash \{e\}, l(\sigma) \leq \epsilon\},$$

$$M_{[\epsilon,\infty)} = \{x \in M | \forall [\sigma] \in \pi_1(M, x), l(\sigma) \geq \epsilon\},$$

where by $[\sigma]$ we mean the element of $\pi_1(M, x)$ corresponding to the (based at x) loop σ.

$M_{(0,\epsilon]}$ will be called the $\epsilon-$thin part of M, while $M_{[\epsilon,\infty)}$ will be called the $\epsilon-$thick part of M.

Theorem 5.24. *Suppose that M is hyperbolic. Up to taking ϵ small enough (depending on M) we have that*

- *$M_{[\epsilon,\infty)}$ is compact,*
- *M is diffeomorphic to the internal part of $M_{[\epsilon,\infty)}$,*
- *each connected component of $M_{(0,\epsilon]}$ (which will be called* cusp*) is diffeomorphic to $V \times [0, +\infty)$ where V is a flat Riemannian manifold. Also, $\pi_1(V)$ naturally injects in $\pi_1(M)$.*

The situation for the general case is similar (see [Eb]). More precisely, in this case M has finitely many (topological) ends, each of which (has a neighborhood U which) is topologically $V \times (0, +\infty)$, where V is a manifold with virtually nilpotent fundamental group. Also, $\pi_1(V)$ injects in $\pi_1(M)$. Finally, the preimage of U under the universal covering map is an equivariant (with respect to $\pi_1(M)$) family of open horoballs, see the next section for the definition.

Let $U_1, \dots, U_k$ be neighborhoods of the cusps chosen as above, and suppose that U_i is homeomorphic to $V_i \times (0, +\infty)$.

Definition 5.25. $M \backslash (U_1 \cup \dots \cup U_k)$ is said to be obtained from M truncating the cusps.

$\pi_1(V_i) < \pi_1(M)$ are called the cusp subgroups of M.

Let us consider once again the case that M is hyperbolic. If $dim(M) = 3$, V is a torus, and we also have the following basic topological informations about $N = M_{[\epsilon,\infty)}$ (see [BP]):

Proposition 5.26.

1. *N has toric boundary,*
2. *N is atoroidal (i.e. each subgroup of $\pi_1(N)$ isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$ is conjugated to a cusp subgroup),*
3. *N is irreducible (i.e. any smoothly embedded S^2 in N bounds a ball),*
4. *N is $\partial-$incompressible (i.e. $\pi_1(\partial N)$ injects into $\pi_1(N)$),*
5. *$\pi_1(N)$ is infinite.*

A part of the geometrization program is to prove that if N is a compact $3-$manifold satisfying (1) through (5), then $\mathring{N}$ admits a hyperbolic metric of finite volume. This has been proved by Thurston in the case case that N is Haken (therefore, in particular, if $\partial N \neq \emptyset$).

A theorem by Thurston shows that plenty of knot complements admits a finite volume hyperbolic metric. To state it, we need to define 2 families of knots.

Definition 5.27. A torus knot is a knot which lies on an unknotted torus.

Definition 5.28. Let K_1 be a non-trivial knot in a solid torus. Let K_2 be a non-trivial knot (in S^3) and h an identification of the solid torus with a tubular neighborhood of K_2. The knot $h(K_1)$ is called satellite knot.

Theorem 5.29. *Let K be a knot which is neither a torus knot or a satellite knot. Then $S^3 \backslash K$ admits a hyperbolic metric of finite volume.*

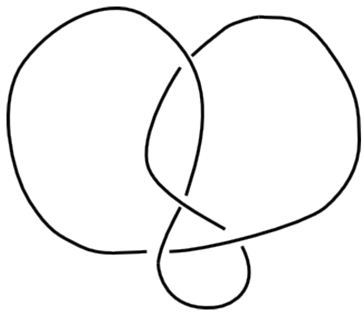


Figure 5.2: Figure 8 knot, the most famous example of hyperbolic knot.

A knot which admits a hyperbolic finite volume structure on the complement is called hyperbolic.

There is also another interesting family of finite volume hyperbolic manifolds in dimension 3 (found by Thurston as well). Consider a surface of finite type S with negative Euler characteristic. Recall that a diffeomorphism $\phi : S \to S$ is pseudo-Anosov if ϕ^n is not isotopic to the identity for each $n \in \mathbb{N}^+$ and if it does not fix the isotopy class of a simple closed curve in S.

Theorem 5.30. *If ϕ is pseudo-Anosov the mapping torus of ϕ admits a hyperbolic structure of finite volume.*

5.4 Busemann functions

We are going to need the notions of Busemann function, horosphere and horoball. For what follows we will assume familiarity with the basic notions of $CAT(k)$ spaces, exposed in [BH]. We will use the same notation as in that book. Let us fix a proper $CAT(-a^2)$ space H, for $a > 0$. The reader who already knows Busemann functions can easily check that the following definition is equivalent to the the usual one.

Definition 5.31. Consider any infinite $\tau \in {}^*\mathbb{R}^+$ (fixed throughout the section). The Busemann function $b_c : X \to \mathbb{R}$ associated to the geodesic ray c is the function defined by

$$b_c(x) = st(d(x, c(\tau)) - \tau).$$

Notice that from this definition and the (transfer principle applied to the) convexity of the metric in H, if follows immediately that Busemann functions are convex (this is well-known, see [BH] or [BGS]).

Definition 5.32. A horosphere based at $x \in \partial H$ is a set of the kind $b_c^{-1}(l)$, for some geodesic ray which represent x and some $l \in \mathbb{R}$. The (closed) horoball (bounded by the horosphere $b_c^{-1}(l)$) is $b_c^{-1}([l,+\infty))$. An open horoball is a set of the kind $b_c^{-1}((l,+\infty))$.

The definition we gave of Busemann function is even closer than the standard one to the idea that a horosphere is a sphere of infinite radius centered in a point infinitely far away.

Recall that for each convex non-empty set A in H and $x \in H$, there exists a unique point $\pi_A(x)$, called the (closest point) projection of x on A, such that $d(x, \pi_A(x)) = d(x, A)$.

Let $S = b_c^{-1}(l)$ and let B the horoball bounded by S. Notice that $\pi_B(H \backslash \mathring{B}) \subseteq S$. Therefore, the projection on B restricted to $H' = H \backslash \mathring{B}$ is usually denoted by π_S and called the projection on the horosphere.

We want to characterize π_S. As H is proper, for each $\xi \in {}^*H$ whose distance from some point of H is finite there exists exactly one element of H, denoted $st(\xi)$, such that $d(\xi, st(\xi)) \ll 1$.

Lemma 5.33. *Consider some $x \in H'$ and let γ_x be the *geodesic starting from $c(\tau)$ passing through x. Then*

$$\pi_S(x) = st(\gamma_x(\tau - l)).$$

Proof. For each $q \in S$, $d(x,q) \geq st(d(x,c(\tau)) - d(c(\tau),q)) = st(d(x,c(\tau)) - \tau)$. Also, $d(x,c(\tau)) = d(x,\gamma_x(\tau-l)) + d(\gamma_x(\tau-l),c(\tau))$ and $st(\gamma_x(\tau-l)) \in S$, therefore $st(\gamma_x(\tau-l))$ is the (only) point in S such that the inequality is actually an equality. $\square$

Remark that the geodesic ray γ given by $\gamma(t) = st(\gamma_x(\tau - t))$ is asymptotic to c.

In the following section, we will need not only that projections on horosphere decrease lengths of paths (this is true for projections on convex sets in any $CAT(0)$ space, see Proposition II.2.4 in [BH]), but also the following (presumably well-known) stronger statement.

Lemma 5.34. *Consider a curve γ which is not contained in the horoball bounded by S and such that $d(\gamma, S) \geq k$. Then $l(\pi_S(\gamma)) \leq e^{-ak} l(\gamma)$.*

Actually, we will apply apply the lemma in the case that H is a Riemannian manifold. The proof under that hypothesis can also be found in [HI, Proposition 4.1].

Proof. Notice that we can first project γ on $S' = b_c^{-1}(l-k)$ (decreasing its length), and then project the obtained curve in S' on S. We can therefore assume $\gamma \subseteq S'$.

Let us show how, using comparison triangles, we can reduce to studying the problem in M_{-a^2} (that is, $\mathbb{H}^n$ with the metric rescaled by $1/a$). Consider points p', q' on S' and let p, q be their projections on S. Consider the *geodesic triangle with vertices $c(\tau), p', q'$. Notice that p has infinitesimal distance from the point p_0 on $[p', c(\tau)]$ such that $d(p', p_0) = k$. Similarly for q and q_0. By the $CAT(-a^2)$ inequality, $d(\overline{p_0}, \overline{q_0}) \leq d(p_0, q_0)$ (notice that $st(d(p_0, q_0)) = d(p, q)$). The description of the projection on a horosphere holds also for M_{-a^2}. Therefore, considering the comparison triangle, $st(d(\overline{p_0}, \overline{q_0}))$ is the distance between the projection on a horosphere $\widetilde{S}$ of points $\tilde{p}$, $\tilde{q}$ with $d(\tilde{p}, \tilde{q}) = d(p', q')$ on a horosphere $\widetilde{S'}$, with $d(\widetilde{S}, \widetilde{S'}) = k$ and $\widetilde{S}$ contained in the horoball bounded by $\widetilde{S'}$.

In order to compute the length of a path we are interested in pairs of points p, q with distance close to 0. The distance between the corresponding points $\tilde{p}, \tilde{q}$ as above (as well as the distance between their projections on $\widetilde{S}$) compares well with the corresponding distance in the path metric of $\tilde{S}$. This tells us that it is enough to prove that $\pi_{\widetilde{S}} : \widetilde{S'} \to \widetilde{S}$ satisfies $d_{\widetilde{S}}(\pi_S(x), \pi_S(y)) \leq e^{-ak} d_{\widetilde{S}}(x, y)$. Indeed, using the upper-half space model it is easy to prove that $\pi_{\widetilde{S}}$ satisfies $d(\pi_S(x), \pi_S(y)) = e^{-ak} d(x, y)$ (we can choose $\widetilde{S'} = \{(x_1, \dots, x_n) | x_n = 1\}$, $\widetilde{S} = \{(x_1, \dots, x_n) | x_n = ak\}$).

□

5.5 $\pi_1(M)$ is relatively hyperbolic

Recall that we fixed M to be a finite volume negatively curved manifold. Let X denote the universal cover of M with truncated cusps. We have that X is quasi-isometric to $\pi_1(M)$ through a quasi-isometry which takes the family of lateral classes of cusp subgroups to an equivariant family of horospheres $\mathcal{H}$. Denote by H the universal cover of M. The distance in X will be denoted simply by d, and the distance in H by d_H.

Let us proceed with a list of the Riemannian geometry lemmas we will need. Let us start with a consequence of the fact that $d(\cdot, B)$ is convex when B is a horoball.

Remark 5.35. A geodesic in H intersects each horosphere in 0,1 or 2 points.

The remark above will be used implicitly a few times.

Suppose that the curvature of M is pinched between $-b^2$ and $-a^2$ ($a > 0$). First of all, by Theorem 1A.6 and Proposition III.1.2 in [BH]:

Lemma 5.36. *H is $CAT(-a^2)$, and in particular it is uniquely geodesic and Gromov-hyperbolic.*

The following lemma is an application of Proposition 3.9.11 in [Kl] (see also Proposition 4.1 in [Fa], where we borrow the notation from).

Lemma 5.37. *Let $\gamma(t)$ be a geodesic line in H and let $\beta : [0, \tau] \to H$ be a curve in H from $\beta(0) = p$ to $\beta(\tau) = q$. Suppose that $d(p, \gamma) = d(q, \gamma) = K$ and that $d(\beta(t), \gamma) \geq K$ for each $t \in [0, \tau]$. Let p' and q' be the projections of p and q respectively on γ. Then*

$$d_H(p', q') \leq l(\beta)e^{-aK}.$$

The consequence we will actually need is:

Corollary 5.38. *Let γ be a geodesic line in H and $\beta : [0, \tau] \to H$ be a curve such that $d(\beta(t), \gamma) \geq K \geq 2/(3a)\log(2)$ for each $t \in [0, \tau]$. Set $p = \beta(0)$ and $q = \beta(\tau)$. Let p' and q' be the projections of p and q respectively on γ. Then*

$$d_H(p', q') \leq l(\beta)e^{-aK/2}.$$

Proof. Set $K' = d(\beta, \gamma) \geq K$. Let δ_1 (resp. δ_2) be the perpendicular from p (resp. q) to γ. Let p'' be the point on δ_1 at distance K' from γ and define similarly $q'' \in \delta_2$. Let β' be obtained by concatenating $[p'', p]$, β and $[q, q'']$. Applying the above lemma to β' we obtain

$$d_H(p', q') \leq l(\beta')e^{-aK'} = (l(\beta) + d(p'', p) + d(q, q''))e^{-aK'} \leq l(\beta)2e^{-aK'},$$

as $d(p, p'') + d(q'', q) \leq l(\beta)$. In fact, if $x \in \beta$ is such that $d(x, \gamma) = K'$, then $d(p, p'') + d(q, q'') \leq d(p, x) + d(x, p) \leq l(\beta)$ (recall that p'', q'' lie on the perpendiculars from p, q to γ). As $K' \geq K$ and $\log(2) - aK \leq aK/2$, we are done.

$\square$

We already stated a stronger statement than the one below, but it is probably convenient to emphasize when just the following one is needed.

Lemma 5.39. *Let β be a path which does not intersect the horoball B bounded by the horosphere S. Let μ be the projection on S of β. Then $l(\mu) \leq l(\beta)$.*

This has an interesting corollary.

Corollary 5.40. *Each $S \in \mathcal{H}$ is geodesic in X.*

This definition and the following lemma are taken from [Fa] (see Lemma 4.4 and the definition above).

Definition 5.41. Let S be a horosphere in H and γ a geodesic line which does not intersect it. Let T_γ be the set of points $s \in S$ such that there exists t with the property that $[s, \gamma(t)] \cap S = \{s\}$. The visual size of S is

$$vs(S) = \sup_{\{\gamma : \gamma \cap S = \emptyset\}} diam(T_\gamma).$$

Lemma 5.42. *There exists $D > 0$ such that for each horosphere S we have $vs(S) \leq D$.*

For later purposes, we assume that we chose to truncate the cusps in such a way that different horospheres in $\mathcal{H}$ have distance at least $12D$.

Corollary 5.43. *If γ is a geodesic from $x \in H$ to $s \in S$, for some $S \in \mathcal{H}$, such that $\gamma \cap S = \{s\}$, then $d(s, \pi_S(x)) \leq D$.*

Proof. Let δ be a d_H-geodesic line perpendicular to γ in x. Notice that, as $d(\cdot, B)$ is convex, when B is the horoball bounded by S, δ does not intersect S. We have that s and $\pi_S(x)$ both belong to T_δ, and therefore by the lemma $d(\pi_S(x), s) \leq D$.

□

Corollary 5.44. *If γ is as in the previous corollary, $diam(\pi_S(\gamma)) \leq 2D$.*

Proof. For each $x_1, x_2 \in \gamma$, $d(s, \pi_S(x_i)) \leq D$, so $d(\pi_S(x_1), \pi_S(x_2)) \leq 2D$.

□

We will also need the following property of projections (Proposition 4.3 in [Fa]):

Lemma 5.45. *Let S and S' be non-intersecting horospheres in H, based at distinct points of ∂H. Then the diameter of $\pi_S(S')$ (measured in the metric d_S) is bounded by $4/a$.*

Putting together this lemma and Corollary 5.43, we obtain the following.

Lemma 5.46. *Suppose that there exists a d_H-geodesic δ from $p \in S$ to S', for some non-intersecting horospheres S and S' based at different points of ∂H, which intersects S, S' only in its endpoints. Then $d_S(p, \pi_S(q)) \leq 4/a + D$ for each $q \in S'$.*

We will have to compare distances on X with corresponding distances in H. The next lemma, which is an application of Theorem 4.6 in [HI], will be sufficient for our purposes.

Lemma 5.47. *There exists an increasing unbounded function $g : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that for each $p, q \in X$ we have*

$$d_H(p, q) \geq g(d_X(p, q)).$$

The last lemma we need is

Lemma 5.48. *There exists a function $C : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ with the following property. Let γ be a d_H-geodesic line and consider $p \in \gamma$. Suppose that p is contained in the horoball bounded by the horosphere S and that $d(p, S) \leq x$, for some $x \geq 0$. Then there exists $q \in \gamma \cap S$ such that $d_H(p, q) \leq C(x)$.*

Proof. Let S' be the horosphere based at the same point of ∂H as S and passing through p. By convexity of the Busemann functions, at least one geodesic ray γ' contained in γ starting from p is external to the horoball bounded by S'. By Corollary 5.44, we have that the projection of γ' on S' has diameter bounded by $2D$. Consider a point q on γ' at a distance $d(p,S)+2D$ from q. Then it is easily seen that $d(S',q) \geq d(p,S) \geq d(S,S')$. This implies that q is on S or external to the horoball bounded by S, so we are done.

□

Now, we are ready to prove the main result of the section.

Theorem 5.49. *$\pi_1(M)$ is hyperbolic relative to its cusp subgroups.*

This result was proven [Fa], using another definition of relative hyperbolicity, which is equivalent to the (first) one we gave as they are both equivalent to yet another definition given by Osin, see [Os1] and [DS].

The remainder of this section is devoted to the proof of this theorem. What we will actually prove is that X is asymptotically tree-graded with respect to $\mathcal{H}$, using the characterization provided by Proposition 5.21. Of course, the functions π_S for $S \in \mathcal{H}$ will be the closest point projections.

Notice that if $S_1 \neq S_2 \in \mathcal{H}$ are distinct, then they do not intersect and they are based at different points of ∂H, therefore $(AP3)$ immediately follows from Lemma 5.45.

Let us now prove property (AP_w2). Consider $x \in X$, $S \in \mathcal{H}$ and set $d = d(x,S)/2$. From Lemma 5.34 and Lemma 5.47, it follows that, for each $y \in B_X(x,d)$, $d(\pi_S(x), \pi_S(y)) \leq de^{-ag(d)}$. As g is unbounded, $e^{-ag(d)} \to 0$ for $d \to +\infty$.

From now on we will have to analyze geodesics and quasi-geodesics in X. As geodesics in H are easier to study, we want to reduce to studying them.

To simplify statements and proofs, let us fix a constant $c \geq 0$ and by "almost-geodesic" we mean continuous $(1,c)-$quasi-geodesic. The constants we will find depend on c.

Remark 5.50. The statements in this section which do not involve "first points" with some property hold also for $(1,c)-$quasi-geodesics (with different constants, though), as each $(k,c)-$quasi-geodesic is at a Hausdorff distance bounded by $k+c$ from a continuous $(k,k+c)-$ quasi-geodesic.

If γ is any path in H, denote by $Sat(\gamma)$ the union of $\gamma \cap X$ and of the horospheres $S \in \mathcal{H}$ which intersect γ.

Lemma 5.51. *Let β be an almost-geodesic in X, and γ be the geodesic in H with the same endpoints. There exists d, not depending on β, such that $\beta \subseteq N^X_d(Sat(\gamma))$. Also, there exists κ such that d can be chosen to be κc for each $c \geq 1$.*

From now on, d will be the constant appearing in this lemma, for the fixed c.

Proof. Fix some $K \geq 2/(3a)\log(2)$. Let $\beta' : [0,\tau] \to X$ be a maximal subgeodesic of β which lies outside $N^H_K(Sat(\gamma))$ and let p, q be its endpoints. Denote by p' and q' the projections of p and q on γ. Let δ be the path in X obtained in the following way:

- let γ' be obtained as the concatenation of $[p'', p']$, where p'' is the last point on $[p, p']$ contained in X, the subgeodesic of γ with endpoints p' and $[q', q'']$, where q'' is defined similarly to p'' ($[p, p']$ and $[q', q]$ are the d_H-geodesics with the corresponding endpoints),
- substitute maximal subpaths of γ' which are contained in a horoball bounded by some $S \in \mathcal{H}$ with a geodesic in S with the same endpoints.

Let n_1 be the number of maximal subpaths of β' which intersects horospheres in S only in their endpoints, and let n_2 be the number of maximal subpaths of β' which are entirely contained in some $S \in \mathcal{H}$. Finally, set $n = n_1 + n_2$. We have that

$$l(\beta') \leq d(p,q) + c \leq 2K + l(\delta) + c \leq 2K + l(\beta')e^{-aK/2} + 2Dn_2 + c,$$

where D is as in Lemma 5.42 (we considered the projection on the geodesic line containing γ). Also, we have that $n_2 - 1 \leq n - 1 \leq l(\beta')/(6D)$, as distance between different horospheres in $\mathcal{H}$ is at least $12D$. If we choose K large enough so that $e^{-aK/2} \leq 1/3$, we have that

$$1/3l(\beta') \leq 2K + 2D + c,$$

and in particular $l(\beta')$ can be bounded by some L, which depends only on H and X. Therefore, we have that $\beta \subseteq N^X_d(Sat(\gamma))$ for $d = K + L/2$. From the estimate we did, it is clear that κ as in the statement exists.

□

The following lemma readily implies $(AP1)$.

Lemma 5.52. *There exists R with the following property. If β is a geodesic in X from $p \in X$ to $s' \in S$, for some $S \in \mathcal{H}$, then $\beta \cap B_X(\pi_S(p), R) \neq \emptyset$.*

Proof. Consider β, p, s', S as in the statement, and let γ be the geodesic in H from p to s'. Let γ' be the initial subgeodesic of γ such that $\gamma' \cap S = \{s\}$, for some $s \in S$. Let $\mathcal{S}$ be the subset of $Sat(\gamma)$ given by points on γ which lie on γ' or on a horosphere $S' \in \mathcal{H}$, $S' \neq S$, which intersect γ'.

Suppose that we are able to find a bound R' for the diameter of $A = N^X_d(\mathcal{S}) \cap N^X_d(S)$. Then, as β intersects A, $s \in A$ and $d(s, \pi_S(x)) \leq D$ (by Corollary 5.43), we have $d(\pi_S(p), \beta) \leq D + R'$. Setting $R = D + R'$, we are done.

We have to find R'. Let u, v be points in A, u', v' points in $\mathcal{S}$ closer than d from u, v. Also, let u'', v'' be points on γ such that u', u'' and v', v'' lie on the same horosphere of $\mathcal{H}$, or $u'' = u'$ (resp. $v'' = v'$) in case $u' \in \gamma$ (resp. $v' \in \gamma$). Keeping into account Lemma 5.39, Lemma 5.45 and Corollary 5.44, we have

$$d(u,v) \leq d(u,\pi_S(u)) + d(\pi_S(u),\pi_S(u')) + d(\pi_S(u'),\pi_S(u'')) + d(\pi_S(u''),\pi_S(v'')) +$$

$$d(\pi_S(v''),\pi_S(v')) + d(\pi_S(v'),\pi_S(v)) + d(\pi_S(v),v) \leq$$

$$d + d + 4/a + 2D + 4/a + d + d = 4d + 8/a + 2d.$$

We can set $R' = 4d + 8/a + 2d$. □

Only one thing is left to check.

Lemma 5.53. *$\mathcal{H}$ is asymptotically transverse-free.*

Proof. It is enough to suppose $c \geq 1$ and to prove that for each almost-geodesic triangle Δ which is $\mathcal{H}-$almost transverse with constants $K = \kappa c, E \geq 1$, we have that Δ is $\lambda(E+c)-$thin, for some λ to be chosen independently from E, c. Let β_i, $i = 0, 1, 2$, be the sides of Δ, and let γ_i be the corresponding d_H-geodesics.

Claim. For each point p on β_i there is a point q on γ_i such that $d(p,q) \leq \lambda_1(E+c)$, for some λ_1 which does not depend on E, c.

Proof. We can assume that the endpoints of β_i have distance at least $2E+2$. By Lemma 5.51, p is either at distance at most κc from a point in $\gamma \cap X$, and this case is fine, or there exists $S \in \mathcal{H}$ with $\gamma \cap S \neq \emptyset$, $d(p, S) \leq \kappa c$. In the last case, by the hypothesis on Δ, a point p' on β_i whose distance from p is $E+1$ does not belong to $N^X_{\kappa c}(S)$. If this point is close at most κc from a point in $\gamma \cap X$, we are done. Otherwise, there exists $S' \in \mathcal{H}$ with $\gamma \cap S' \neq \emptyset$ and $d(p', S') \leq \kappa c$. Let γ' be the subgeodesic of γ which intersects S (resp. S') only in one of its endpoints q (resp. q'). Let $r' \in S'$ be a point such that $d(r', p') \leq \kappa c$. By keeping into account Corollary 5.43, Lemma 5.45 and the fact that π_S decreases distances, we get

$$d(q,p) \leq d(q,\pi_S(q')) + d(\pi_S(q'),\pi_S(r')) + d(\pi_S(r'),\pi_S(p')) + d(\pi_S(p'),\pi_S(p)) +$$

$$d(\pi_S(p),p) \leq D + 4/a + \kappa c + (E+1) + \kappa c = D + E + 2\kappa c + 4/a + 1.$$

□

Claim. For each point r on $\gamma_i \cap X$ there is a point s on β_i such that $dr, s) \leq \lambda_2(E+c)$, for some λ_2 which does not depend on E, c.

Proof. Set $\gamma = \gamma_i$ for simplicity and let p, q be its endpoints. Let γ' be a maximal subsegment of γ contained in X such that $r \in \gamma'$. Let p' and q' be the endpoints of γ'. Set $K = 9d + 12/a + 3D + 1$ (notice that there exists λ_2 such that $K \leq \lambda_2(E + c)$). We can assume that $d(r, p), d(r, q) > K$. Let p'' be the point between p and p' at a distance K from p. Define q'' in an analogous way. Let Sat_p (define Sat_q similarly) be the subset of $Sat(\gamma)$ of points on γ before p'' or on a horosphere which intersects γ before p''. Notice that $p \in Sat_p, q \in Sat_q$. We have that $Sat(\gamma) \backslash B_X(r, K - d) \subseteq Sat_p \cup Sat_q$. We want to prove that $(N_d^X(Sat_p) \cup N_d^X(Sat_q)) \backslash B_X(r, K)$ is not connected, which easily implies the thesis as β_i is a continuous path which starts in Sat_p and ends in Sat_q contained in $N_d^X(Sat(\gamma))$.

Consider, by contradiction, some $t \in N_d^X(Sat_p) \cap N_d^X(Sat_q) \backslash B_X(r, K)$. Let t_p and t_q be points on, respectively, Sat_p and Sat_q such that $d(t, t_p), d(t, t_q) \leq d$. We have that t_p and t_q cannot both lie on γ, as $d(t_p, t_q) \leq 2d$. Suppose that t_p lies on some $S \in \mathcal{H}$ which intersect γ before p'' in u. Using Lemma 5.46 (there are 2 cases to consider), we obtain $d(t_q, u) \leq d(t_q, \pi_S(t_q)) + d(\pi_S(t_q), u) \leq 2d + 4/a + D$. If $d(u, r) \leq 6d + 8/a + 2D$ we have $d(t_q, r) \leq 8d + 12/a + 3D = K - d - 1$, a contradiction. Therefore $d(u, r) > 6d + 8/a + 2D$. If $t_q \in \gamma$, we directly have $d(t_q, u) \geq d(r, u) > 6d + 8/a + 2D$, a contradiction. On the other hand, if t_q lies on a horosphere S' intersecting γ in v, we have $d(v, u) \geq d(r, u) > 6d + 8/a + 2D$. In the same way we obtained $d(t_q, u) \leq 2d + 4/a + D$, we can get $d(t_p, v) \leq 2d + 4/a + D$. Therefore

$$d(u, v) \leq d(u, t_q) + d(t_q, t_p) + d(t_p, v) \leq 6d + 8/a + 2D,$$

a contradiction.

□

We are ready to conclude the proof. Consider a point p on β_i. There exists a point $q \in \gamma_i$ whose distance from p is at most $\lambda_1(E + c)$. Let δ be a hyperbolicity constant for H. There exists a point r' on either γ_{i+1} or γ_{i-1} (suppose $r' \in \gamma_{i+1}$) such that $d_H(q, r) \leq \delta$. By Lemma 5.48, there exists a point r on $\gamma_{i+1} \cap X$ such that $d(p, r) \leq C(\delta) + \delta$. By the second claim, there exists a point s on β_{i+1} such that $d(r, s) \leq \lambda_2 c$. Putting all this together, we get

$$d(p, s) \leq (\lambda_1 + \lambda_2)(E + c) + C(\delta) + \delta.$$

Hence, it is clear that we can choose λ large enough so that Δ is $\lambda(E + c)-$thin, as required.

□

Chapter 6

Asymptotic cones of relatively hyperbolic groups

Throughout the chapter G will denote a group which is hyperbolic relative to its subgroups $H_1, \dots, H_n$. To avoid trivialities, let us assume that each H_i has infinite index in G and is infinite. We also fix a finite system of generators S. We will often identify G and its image in the Cayley graph associated to S. With an abuse, we will denote by e the neutral element of G and also its projection on an asymptotic cone of G.

6.1 Hyperbolic elements

In this section we study some algebraic features of relatively hyperbolic groups and their interaction with the geometry of the asymptotic cones. Most purely algebraic results in this section already appear in [Os1] or [Os2].

Definition 6.1. A hyperbolic element of G is an element which is not conjugated to any H_i.

Remark 6.2. Notice that for each subgroup H of G, ${}^*H \cap G = H$. This depends on the fact that balls in G contain finitely many points.

Proposition 6.3. *There are finitely many conjugacy classes of hyperbolic elements of finite order. In particular there are finitely many possible orders of hyperbolic elements.*

Proof. It is enough to prove that if $g \in {}^*G$ is *hyperbolic and $d(e, g)$ is infinite but minimal in the conjugacy class of g, then the order of g is not *finite. In fact, this implies that if $g \in G$ is hyperbolic and each element in its conjugacy class is far enough from e, then the order of g is infinite. For each conjugacy class of hyperbolic elements of finite order we can then find a representative in a ball of fixed radius, which contains finitely many elements.

Therefore, consider g as above and set $d = d(e, g)$. Our aim is to prove:

Lemma 6.4. *Up to substituting g with an element in its conjugacy class, we have*

$$d(g^{\rho+1}, e) \geq d(g^\rho, e) + d/2,$$

for each $\rho \in {}^*\mathbb{N}$.

Proof. Consider a geodesic γ in X from e to $[g]$, obtained projecting a *geodesic β from e to g. Let us start by proving that γ and $g\gamma$ intersect only in $[g]$. Suppose that this is not the case and that $g\gamma$ intersects γ in, say, $p = [\beta(t)] \neq [g]$ for some $t \in {}^*\mathbb{N}$ (so that $\beta(t) \in {}^*G$). Then we have that, for some $u \in {}^*\mathbb{N}$, $\epsilon = d(g\beta(u), \beta(t)) \in o(d)$. We therefore have

$$d(\beta(u)^{-1}g\beta(u), e) \leq d(\beta(u)^{-1}g\beta(u), \beta(u)^{-1}\beta(t)) + d(\beta(u)^{-1}\beta(t), e) =$$

$$\epsilon + |t - u| < d,$$

which contradicts the minimality of $d(e, g)$.

Notice that so far we have that γ and $g\gamma$ satisfy conditions (1) and (2) in the definition of concatenating well. To get condition (3), we will consider 3 cases.

1) The first case is that γ is contained in a piece P. Then g can be written as k_1hk_2 for some $h \in H_i$, $d(k_i, e) \in o(d)$ (P contains e and $[g]$). We have that P, which is induced by $k_1\,{}^*H_i$, is different from gP, which is induced by $gk_1\,{}^*H_i$, by the fact that $k_1^{-1}gk_1 \notin {}^*H_i$. Therefore we get condition (3).

Now, consider the geodesic (or geodesic ray) α induced by a *geodesic from e to g^ρ in $X = C(G, g^\rho, d)$. Set $\gamma' = g^{\rho-1}\gamma$. As γ' and $g\gamma'$ concatenate well, by Remark 4.12 We have that α and γ' or α and $g\gamma'$ concatenate well, and so, using Lemma 5.10, we get that either $d(g^{\rho-1}, e) \geq d(g^\rho, e) + d - \epsilon > d(g^\rho, e) + d/2$ or $d(g^{\rho+1}, e) \geq d(g^\rho, e) + d/2$. By induction, we can exclude the first case.

If γ is not contained in any piece, let p be the first point on γ such that there exists a piece P containing p and $[g]$, and let γ' be the (non-trivial) initial subpath of γ which ends in p. Also, let q be the last point on $g\gamma$ such that $q \in P$, and let γ'' be the (non-trivial) final subpath of γ starting from $g^{-1}q$.

2) Suppose that $p = q$. In this case we necessarily have $p = q = [g]$, and also γ and $g\gamma$ concatenate well. The conclusion follows just as in case 1.

3) The final case is that $p \neq q$. In this case, consider $g'' \in {}^*G$ which projects on p. We have that g can be written as $g''k_1hk_2$, where $h \in H_i$, $d(k_1, e), d(k_2, e) \in o(d)$. Set $g' = (hk_2)g(hk_2)^{-1}$. By minimality of $d(e, g)$ in the conjugacy class of g, we have that $d(e, g') \geq d$. Let δ_1 (resp. δ_2) be the subpath of γ from p to $[g]$ (resp. from e to p). Let δ be the path from e to $[g']$ obtained as the concatenation of $(g''k_1)^{-1}\delta_1$ and $hk_2\delta_2$. Notice that

$l(\delta) = l(\gamma) = 1 \leq st(d(e, g')/d)$, and therefore δ is a geodesic from e to $[g']$ and $st(d(e, g')/d) = 1$. We want to show that δ and $g'\delta$ concatenate well. Condition (1) is clear. Both conditions (2) and (3) follow from these facts:

- δ_2 is not trivial as γ is not contained in P,
- an initial subpath of $g'\delta$ is contained in $g'(g''k_1)^{-1}P = hk_2P$, and
- from $\delta_2 \cap P = \{p\}$ we get $hk_2\delta_2 \cap hk_2P = \{hk_2p\} = \{[g']\}$.

Up to substituting g with g', the conclusion follows from the same argument as in case (1).

□

The lemma clearly implies that the distances from e of the powers of g are not bounded, therefore the order of g cannot be *finite.

□

Corollary 6.5. *If g is a hyperbolic element of infinite order, then $\langle g \rangle$ is quasi-isometrically embedded in G.*

Proof. Consider some $g_0 \in {}^*G$ which is *hyperbolic of *infinite order, and which has the additional property that each element in its conjugacy class has infinite distance from e. An inductive argument based on Lemma 6.4 implies that there exists some element g in the conjugacy class of g_0 with the property that for each $\rho_1, \rho_2 \in {}^*\mathbb{Z}$

$$d(g^{\rho_1}, g^{\rho_2}) = d(e, g^{\rho_2 - \rho_1}) = d(e, g^{|\rho_2 - \rho_1|}) \geq |\rho_2 - \rho_1| d/2,$$

for some $d > 0$. As $d(g^{\rho_1}, g^{\rho_2}) \leq |\rho_2 - \rho_1| d(e, g)$, we have that $\langle g \rangle$ is *quasi-isometrically embedded in G. This implies that $\langle g_0 \rangle$ is also *quasi-isometrically embedded, as conjugation is an isomorphism of $^*G$ and therefore a *quasi-isometry.

The above considerations imply that there exists $r_0 \in \mathbb{R}$ such that if $g_0 \in G$ is hyperbolic of infinite order, and each element in its conjugacy class has distance greater than r_0 from e, then $\langle g_0 \rangle$ is quasi-isometrically embedded in G. The conclusion follows from the easy lemma below, considering a power of the hyperbolic element of infinite order g high enough so that the additional condition above holds. In fact, if $n \mapsto (g^m)^n$ ($m > 0$) is a quasi-isometric embedding, then also $n \mapsto g^n$ is a quasi-isometric embedding.

□

Lemma 6.6. *If there exist $n \in \mathbb{N}^+$ and $g \in G$ such that $g^n \in H_i$ and g has infinite order, then $g \in H_i$. In particular, powers of hyperbolic elements are hyperbolic.*

Proof. Consider any asymptotic cone X of G with basepoint e and let P be the piece induced by ${}^*H_i$. We have that $gP \cap P$ contains infinitely many points (projections of powers of g). Therefore $gP = P$, which implies $g \in {}^*H_i$, whence $g \in H_i$.

□

Remark 6.7. Notice that the proof of Lemma 6.4 contains these intermediate steps:

- in the conjugacy class of g there exists g' such that in the asymptotic cone with basepoint e and scaling factor $d(e, g')$, γ and $g'\gamma$ concatenate well, where γ is a geodesic from e to $[g']$,
- if g' is as above, $d(g'^{\rho+1}, e) \geq d(g'^{\rho}, e) + d(e, g')/2$ for each $\rho \in {}^*\mathbb{N}$.

The above remark motivates the following.

Proposition 6.8. *Suppose that $g \in {}^*G$ is such that in the asymptotic cone with basepoint e and scaling factor $d(e, g)$, γ and $g\gamma$ concatenate well, where γ is a geodesic from e to $[g]$. Then g is *hyperbolic of *infinite order.*

Proof. By Remark 6.7 (and the last phrase in the proof of Proposition 6.3), g has *infinite order.

Suppose that $x^{-1}gx \in {}^*H_i$ and, up to changing x, that $d(g, x\,{}^*H_i) = d(e, x)$. Suppose that $d(e, x) \in o(d)$. Then we would have that the integer powers of g projects on X on a certain piece P. Let $\tilde{\gamma}$ be the geodesic ray obtained concatenating $\gamma, g\gamma, \dots$. We have that P contains $\tilde{\gamma}$, but this is impossible because γ and $g\gamma$ concatenate well.

Suppose instead that $d \equiv d(e, x)$. In this case we have that the geodesic ray $\tilde{\gamma}$ stay at bounded distance from the piece P induced by $x\,{}^*H_i$. But this implies that $\tilde{\gamma}$ contains a geodesic subray contained in P. This is a contradiction as distances of projections of powers of g from P are bounded from below by $st(d(e, x)/d) > 0$.

Finally, suppose that $d \ll d(e, x)$. Consider the asymptotic cone Z with basepoint e and scaling factor $d(e, x)$. Notice that $\langle g \rangle$ is *quasi-isometrically embedded in ${}^*G$ with constants in $O(d(e, g))$ by the first part of the proof of Corollary 6.5. This implies that there is a bilipschitz ray δ in Z whose points are projections of powers of g. We have that the distance between δ and the piece induced by $x\,{}^*H_i$ is bounded, even though points on δ all have positive distance from P. Hence we can find a contradiction similar to the one of the previous case.

□

Proposition 6.8 allows us to find many hyperbolic elements in G. For example:

Corollary 6.9. *Consider some i and $k \in G \backslash H_i$. Then there exists a finite set $C_i \subseteq H_i$ such that if $h \in H_i \backslash C_i$, then hk is a hyperbolic element of infinite order.*

Proof. It is enough to prove that if $h \in {}^*H_i$ and $d(h, e)$ is not finite, then hk is hyperbolic of *infinite order. In fact, this implies that if $h \in H_i$ is sufficiently far from e, then hk is hyperbolic of infinite order.

But if $h \in {}^*H_i$ and $d(h, e)$ is infinite, it is easily seen that hk satisfies the hypotheses of Proposition 6.8.

□

Remark 6.10. One may find many corollaries as above, by changing the kind of elements considered. For example, one may consider elements of the kind $hkh'k'$ for $h, h' \in H$ and $k, k' \in G \backslash H_i$, and show that there exists a finite set $D_i \subseteq H_i$ such that $hkh'k'$ is hyperbolic of infinite order if $h, h' \in H_i \backslash D_i$.

We are now going to show an interesting relation between hyperbolic elements of infinite order and transversal trees.

Lemma 6.11. *Suppose that $g \in G$ is hyperbolic of infinite order. Then in each asymptotic cone with basepoint e the powers of G induce a line contained in T_e.*

Proof. We want to show that the bilipschitz path γ induced by $\nu \in {}^*\mathbb{Z} \mapsto g^\nu$ in X (see Corollary 6.5) is contained in T_e. If this was not the case, in fact, we would have a piece P, induce by, say, $k\,{}^*H_i$, such that $\gamma \cap P$ is a non-trivial subpath. We have that $g\gamma = \gamma$, and in particular $gP \cap P$ contains a non-trivial path. Therefore $gP = P$, which implies $gk\,{}^*H_i = k\,{}^*H_i$ and so $k^{-1}gk \in H_i$, in contradiction with g being hyperbolic.

□

Definition 6.12. We will call the line as above the powers line of g.

From now on, g will denote a hyperbolic element of G of infinite order. All the asymptotic cones will be implied to have basepoint e if not stated otherwise. Denote by $E(g)$ the subset of G of the elements k with the property that there exists an asymptotic cone X such that the action of k on X stabilizes the powers line of g.

We are going to prove that requiring the above property for one asymptotic cone is equivalent to requiring it for all the asymptotic cones with smaller scaling factor, which will easily imply that $E(g)$ is a group. Let us start with a technical lemma.

Lemma 6.13. *There exists $K \in \mathbb{R}^+$ such that for each $\rho_1 < \rho_2$ each *geodesic $\hat{\gamma}$ from g^{ρ_1} to g^{ρ_2} has Hausdorff distance bounded by K from $\{g^{\rho_1}, \dots, g^{\rho_2}\}$.*

Proof. First, it is enough to fix $\rho_1 < \rho_2$ and find $K(\rho_1, \rho_2) \in \mathbb{R}^+$ with the required property for ρ_1, ρ_2. Once we have this, we also have that we can choose K as in the statement. In fact, consider the internal map which associate to each pair $\rho_1 < \rho_2$ the least integer which can be chosen as $K(\rho_1, \rho_2)$. The range of this map is an internal subset of ${}^*N$ contained in $\mathbb{N}$, and therefore it is a finite set.

Fix $\rho_1 < \rho_2$. Consider the *quasi-geodesic $\hat{\beta} = \hat{\beta}_{\rho_1,\rho_2}$ obtained concatenating *geodesics connecting consecutive powers of g with exponents from ρ_1 to ρ_2. Let $\hat{\gamma}$ be as in the statement. If K is such that $\hat{\gamma} \nsubseteq N_K(\hat{\beta})$ does not exist, we can choose (considering the point of maximum distance between $\hat{\beta}$ and $\hat{\gamma}$) an asymptotic cone X, not necessarily with basepoint e, such that

- $\hat{\beta}$ induces in X a bilipschitz path or ray or line β contained in a transversal tree (see Lemma 6.11),
- there is a subpath γ of the path induced by $\hat{\gamma}$ which is either:
  1. a geodesic connecting 2 points on β, but with a point outside β,
  2. a geodesic ray starting from a point on β, contained in the 1-neighborhood of β and containing a point outside β,
  3. a geodesic which does not intersect β, but contained in the 1-neighborhood of β.

All 3 cases are impossible. In fact, case (1) is impossible for arcs connecting 2 points in a transversal tree are contained in the transversal tree. For what regards cases (2) and (3), notice that transversal trees can be added to the set of pieces, obtaining another set of pieces for X (see Remarkadding-trantree:rem). So, by Corollary 4.5 (and the argument within the proof), in both those cases we would have that γ is contained in the same transversal tree as β. But the configurations described cannot arise in a real tree.

A very similar argument proves that some K such that $\hat{\beta} \subseteq N_K(\gamma)$ must exist.

□

Corollary 6.14. *There exists $K \in \mathbb{R}^+$ and a *geodesic line γ such that the Hausdorff distance between γ and ${}^*\langle g \rangle \subseteq {}^*G$ is at most K.*

Proof. For each $\rho \in {}^*\mathbb{N}^+$ choose internally a (reparametrized) *geodesic γ_ρ from $g^{-\rho}$ to g^ρ such that its domain contains 0 and $d(\gamma_\rho(0), e) \leq K$. As $\mathcal{CG}_S(G)$ is proper, any sequence of (reparametrized) geodesics starting from points in a ball of fixed radius has a convergent subsequence, by Ascoli-Arzelà Theorem. Therefore, there is a *subsequence of $\{\gamma_\rho\}$ which *converges to some γ. It is easily seen that γ satisfies the required properties.

□

Corollary 6.15. *If k stabilizes the powers line of g in the asymptotic cone with scaling factor ν, then it does so in each asymptotic cone Y with scaling factor $\nu' \leq \nu$.*

Proof. If k does not stabilize the powers line δ of g in Y, let $[g^{\mu_1}] = [kg^{\mu_2}]$ be the last point on δ lying in $\delta \cap k\delta$. Let γ_1, γ_2 be 2 *geodesic rays contained in γ as in the previous corollary such that $\gamma_1 \cup \gamma_2 = \gamma$ and their common starting point is $\gamma(0)$, chosen in such a way that $d(\gamma(0), e) \in o(\nu')$. Notice that $\theta_i = g^{\mu_1}\gamma_i \subseteq N_K({}^*\langle g \rangle)$. Similarly, $\lambda_i = kg^{\mu_2}\gamma_i \subseteq N_K(k\,{}^*\langle g \rangle)$. The projection of one between λ_1 and λ_2 intersects δ just in $[g^{\mu_1}]$. Suppose it is λ_1. Then λ_1 and θ_i satisfy the hypothesis of Corollary 5.11, for $i = 1, 2$. This is easily shown to imply that for $\nu \geq \nu'$, k does not stabilize the powers line of g in the asymptotic cone with scaling factor ν. In fact, let X be the asymptotic cone with scaling factor ν. We can find ρ such that kg^{ρ} projects on the same point p as $k\gamma(\nu)$. An easy application of Corollary 5.11 (or Lemma 4.9 if $\nu \in O(\nu')$) and the fact that powers of g are closer than $K \in o(\nu)$ from γ, shows that the distance of p from the powers line of g is at least 1.

□

Corollary 6.16. *$E(g)$ is a subgroup of G, which contains $\langle g \rangle$.*

Proof. $\langle g \rangle \subseteq E(g)$ is obvious. If $k_1, k_2 \in E(g)$ stabilize the powers line in the asymptotic cones with scaling factor ν_1, ν_2, then k_1k_2 stabilizes the powers line in the asymptotic cone with scaling factor $\min\{\nu_1, \nu_2\}$.

□

Lemma 6.17. *$\langle g \rangle$ has finite index in $E(g)$.*

Proof. We want to show that the representative k of a right lateral class ${}^*\langle g \rangle k$ of ${}^*\langle g \rangle$ in ${}^*E(g)$ which minimizes the distance from e, is at finite distance from e. This implies the thesis, for the "finite distance" can be chosen to be uniformly bounded.

Suppose that this is not the case. Consider k as above such that $d = d(k, e) \gg 1$. Also, for each infinite ρ, we can choose $k < \rho$ (we will use this later). Consider the asymptotic cone X with scaling factor d and let β be a geodesic in X from e to $[k]$. Let δ be the powers line of g in X. The minimality property of k easily implies that $\beta \cap \delta = \{e\}$. As δ is contained in a transversal tree, this is enough to prove that β and δ concatenate well. Also, if γ_1 and γ_2 are the *geodesic rays contained in γ as in Corollary 6.14 starting from $\gamma(0)$ (chosen to be closer than K from e), then $k\gamma_i$ and β concatenate well for some $i \in \{1, 2\}$ (by Remark 4.12). Corollary 5.11 tells us that $d(k\gamma_i(t), \gamma) > t/2$ for $t \gg d$, and in particular k does not stabilize the powers line of g in any asymptotic cone with scaling factor $t \gg d$.

Fix some infinite t such that all elements of $E(g)$ (which is countable) stabilize the powers line of g in the asymptotic cone with scaling factor t (see

Corollary 6.15). We have that $d(k\gamma(t), \gamma) > t/2$ or $d(k\gamma(-t), \gamma) > t/2$ for each k as above such that $d(e, k) < \sqrt{t}$. This must therefore hold for some k at finite distance from e, which therefore belongs to $E(g)$, in contradiction with our choice of t.

□

Corollary 6.18. *If $k \in G$ stabilizes the powers line of g in one asymptotic cone, then it does so in* each *asymptotic cone.*

Proof. In view of the previous lemma, the powers line of g is the same as the set induced by ${}^*E(g)$. By definition of $E(g)$, if k is as in the statement $k \in E(g) \subseteq {}^*E(g)$. In particular $k\,{}^*E(g) = {}^*E(g)$, and therefore k stabilizes the set induced by ${}^*E(g)$ in each asymptotic cone.

□

Proposition 6.19. *G is hyperbolic relative to $H_1, \dots, H_n, E(g)$.*

Proof. It is enough to prove that the asymptotic cones with basepoint e are tree-graded with respect to the collection of sets induced by *lateral classes of ${}^*H_1, \dots, {}^*H_n, {}^*E(g)$. Property (T_2) is clear, as it holds for a smaller set of pieces. A *lateral class of ${}^*E(g)$ induce a line γ contained in a transversal tree (with respect to the other set of pieces). Therefore $\gamma \cap P$ contains at most one point, if P is induced by some $k\,{}^*H_j$. It remains to prove that the intersection of two lines induced by $k'\,{}^*E(g) \neq {}^*E(g)$ and ${}^*E(g)$ contains at most one point. Suppose that this is not the case. Let X be an asymptotic cone with basepoint e and scaling factor ν such that, if P is the set induced by ${}^*E(g)$, there is a non-trivial geodesic γ contained in $k'P \cap P$. Up to multiplying k' on the left by an element of ${}^*E(g)$ we can assume that e is in the internal part of γ and that there is an element $k \in k'\,{}^*E(g)$ such that $d(k, e) = d(k, {}^*E(g)) \in o(\nu)$. We can find $\rho \in {}^*\mathbb{N}$ such that:

- $d(g^\rho, e) \equiv \nu$
- for each $\mu \in [-\rho, \rho]$ there exists μ' such that $d(g^\mu, kg^{\mu'}) \in o(\nu)$.

The second point is equivalent to: there exists an infinitesimal ξ such that for each $\mu \in [-\rho, \rho]$ there exists μ' with the property that $d(g^\mu, kg^{\mu'}) \leq \xi\nu$. Consider some infinitesimal η such that $\eta \gg \max\{\xi, 1/\nu, d(k, e)/\nu\}$. We have that k stabilizes the powers line of g in the asymptotic cone of G with basepoint e and scaling factor $\eta\nu \gg 1$. If $d(k, e) \in O(1)$ this implies that $k \in E(g)$, a contradiction since we assumed $k\,{}^*E(g) = k'\,{}^*E(g) \neq E(g)$. If $d(k, e)$ is infinite we will reach a contradiction by an argument we used in Lemma 6.17. Consider the asymptotic cone Y with scaling factor $d = d(k, e)$. Let β be a geodesic in Y from e to $[k]$. By our choice of k, and in particular the fact that e is the closest point from k in ${}^*E(g)$, β intersects the powers line of g only in e. This was the property used in the proof of Lemma 6.17

to show that k does not stabilize the powers line of g in any asymptotic cone with scaling factor $t >> d$. But we can choose $t = \eta\nu$, a contradiction.

□

6.2 Pieces containing a fixed point and valency of transversal trees

By X we will denote the asymptotic cone of G with basepoint $e \in {}^*G$ and scaling factor ν. We have that X is asymptotically tree-graded with respect to the set of pieces $\mathcal{P} = \mathcal{P}_1 \cup \cdots \cup \mathcal{P}_n$, where elements of $\mathcal{P}_i$ are induced by *lateral classes of ${}^*H_i$.

Let us start with counting how many pieces contain a fixed point.

Lemma 6.20. *For each $i \in \{1, \dots, n\}$ and $x \in X$, $P(i, x) = \{P \in \mathcal{P}_i | x \in P\}$ has cardinality $2^{\aleph_0}$.*

Proof. As X is homogeneous through isometries which preserve the pieces, it is enough to determine the cardinality of $P(i, e)$. Consider the function $f : \mathbb{N} \to \mathbb{N}$ such that $f(n)$ is the number of lateral classes H_i which have a representative closer than n to e. We have that f is of course increasing an unbounded. In particular, for each infinite $\xi \in {}^*\mathbb{N}$, $f(\xi)$ is an infinite number. Let us fix an infinite $\xi \in o(\nu)$. The *lateral classes counted by $f(\xi)$ give distinct elements of $P(i, e)$, so $|P(i, e)| \geq 2^{\aleph_0}$. Also $|X \backslash \{e\}| \leq 2^{\aleph_0}$ and, as different pieces can intersect in at most one point and each piece contains infinite points, $|P(i, e)| \leq |X \backslash \{e\}|$ (for each $P \in P(i, e)$ consider a point in P different from e). So, we obtained the inequality $|P(i, e)| \leq 2^{\aleph_0}$, and hence the thesis.

□

Let us now study the transversal trees. Notice that they are isomorphic homogeneous trees, so we only need to study the valency of T_e in e.

Proposition 6.21. *The valency of T_e in e is $2^{\aleph_0}$.*

Proof. Notice that $E(g)$ clearly has infinite index in G. Also, the powers line of g is exactly the subset of X induced by ${}^*E(g)$ by Lemma 6.17. Therefore, by Proposition 6.19 and Lemma 6.20, we have that T_e contains $2^{\aleph_0}$ geodesic lines. As the valency of T_e cannot be more than $|T_e| \leq |X| \leq 2^{\aleph_0}$, it must be exactly $2^{\aleph_0}$.

□

6.3 Geodesics in tree-graded spaces

We are going to need some results about the structure of geodesics in tree-graded spaces. Throughout the section $\mathbb{F}$ will denote a tree-graded space with respect to the collection of proper subsets $\mathcal{P}$. Unfortunately, it is not true that all geodesics in $\mathbb{F}$ are obtained by concatenation of geodesics in transversal trees or pieces, as shown by the "fractal" geodesics used in the proof of Lemma 6.11 in [DS]. We want to analyze how far this is from being true.

Remark 6.22. If $\mathbb{F}$ is tree-graded with respect to $\mathcal{P}$, then it is tree-graded also with respect to the collection of subsets $\mathcal{P}'$ obtained from $\mathcal{P}$ by adding a collection of disjoint transversal trees which cover $\mathbb{F}$. When $\mathbb{F}$ is considered as a tree-graded space with respect to $\mathcal{P}'$, all its transversal trees are trivial.

The above remark tells us how we can reduce to studying tree-graded spaces with trivial transversal trees. Henceforth, let $\mathbb{F}$ be such a tree-graded space.

Definition 6.23. Let $\gamma : [0, l] \to \mathbb{F}$ be a geodesic.

- A piece interval is an interval $[a, b) \subseteq [0, l]$ (with $a < b$) such that $\gamma([a, b))$ is contained in a piece and $[a, b]$ is a maximal interval with this property.
- The piece subset P_γ is the union of all piece intervals.

Remark 6.24. A maximal interval I such that $\gamma(I)$ is contained in a certain piece is closed because pieces are closed in $\mathbb{F}$.

Remark 6.25. By the fact that different pieces intersect in at most one point, different piece intervals are disjoint.

It is not true that, for each geodesic $\gamma : [0, l] \to \mathbb{F}$, P_γ is the entire $[0, l)$, however:

Lemma 6.26. $\overline{P_\gamma} = [0, l]$.

Proof. We have that if $x \in [0, l] \backslash \overline{P_\gamma}$ then x is contained in some open interval I such that no non-trivial interval $I' \subseteq I$ has the property that $\gamma(I')$ is contained in just one piece. We have that $\gamma(I)$ is contained in a transversal tree (by Corollary 4.4), a contradiction since transversal trees are trivial. □

The following two definitions are given in order to capture the properties of a geodesic in a tree-graded space with trivial transversal trees. For short, we will call closed-open interval an interval closed on the left and open on the right.

Definition 6.27. An almost filling of an interval $[l, m]$ is a collection $\{I_a\}_{a \in A}$ of non trivial closed-open intervals in $[l, m]$ (in particular A is at most countable) such that

1. if $a \neq a'$, I_a and $I_{a'}$ are disjoint,

2. $\bigcup_{a \in A} I_a$ is dense in $[l, m]$.

Before giving the next definition, let us describe the idea behind it. A P-geodesic is something which wants to keep track of the following data:

- the kind of pieces a certain geodesic γ enters,

- the maximal intervals of the domain of γ mapped in a piece (the $\overline{I_a}$'s, for I_a as below),

- the last point on $\gamma \cap P$ for some P which γ enters ($\Gamma(t)$ for any $t \in I_a$ and the appropriate I_a).

More precisely, it is the associated almost filling that keeps track of the first and second kind of information.

Definition 6.28. Suppose we are given a family of pointed metric spaces $\{(P_i, r_i)\}_{i \in I}$. A P-geodesic Γ with associated almost filling $\{I_a\}_{a \in A}$ of an interval $[l, m]$ and range $\{(P_i, r_i)\}_{i \in I}$ is a function $\Gamma : \bigcup I_a \to \bigsqcup P_i$ such that

1. $\Gamma|_{I_a}$ is constant for each $a \in A$,

2. denoting by $h_\Gamma : \bigcup I_a \to I$ the function such that $\Gamma(t) \in P_i \iff h_\Gamma(t) = i$, we have $d(r_{h_\Gamma(t)}, \Gamma(t)) = l(I_a)$.

The function h_Γ will be called the index selector for Γ.

We could equivalently define Γ as a function with domain A. The reason we chose this definition is merely technical.

Suppose now that $\mathbb{F}$ is a homogeneous tree-graded space such that each piece is homogeneous (we still assume that transversal trees are trivial). Let $\{P_i\}$ be a choice of representatives of isometry classes of the pieces. For each i, fix a basepoint $r_i \in P_i$.

Suppose that for each pair (x, P), where P is a piece and x is a point contained in P, we have a fixed identification of P with some P_i such that x corresponds to r_i. Finally, fix a basepoint $p \in \mathbb{F}$. Given this data, we can associate to each geodesic γ in $\mathbb{F}$ parametrized by arc length a P-geodesic.

Lemma 6.29. *If $\gamma : [0, l] \to \mathbb{F}$ is a geodesic in $\mathbb{F}$ parametrized by arc length, then:*

1. *The collection $\mathcal{I}_\gamma = \{I_a = [q_a, q'_a)\}_{a \in A_\gamma}$ of all maximal closed-open subintervals J of $[0, l]$ such that $\gamma|_J$ is contained in one piece is an almost filling of $[0, l]$.*

2. *Consider the function $h_\Gamma : \bigcup I_a \to I$ which associates to each t the only $i \in I$ such that $\gamma|_{I_a}$ is contained in a piece isometric to $P_{h(t)}$, where $t \in I_a$. Also, let $\Gamma : \bigcup I_a \to \bigsqcup P_i$ be such that $\Gamma(t)$ is the point identified with $\gamma(q'_a)$ under the identification of $(\gamma(q_a), P)$ with $(p_{h_\Gamma(a)}, P_{h_\Gamma(a)})$, where P is the piece which contains $\gamma|_{I_a}$. Then Γ is a P-geodesic and h_Γ is its index selector.*

3. *$\{I_a\}$ and Γ depend only on the endpoints of γ.*

Definition 6.30. Γ as above will be called the P-geodesic induced by γ.

Proof. Lemma 6.26 implies (1), and (2) is clear.

In order to prove (3), we will prove that if γ, γ' are geodesics from p to q and γ intersect the piece P in a non trivial arc, entering it in x and leaving from y, then γ' enters P in x and leave it from y as well.

First of all, we have to prove that γ' intersects P. If this is not the case, then $(\gamma \cup \gamma')\backslash P$ is connected. But the projection of $\gamma\backslash P$ on P consists of 2 points, and the projection of γ' on P consists of one point, as $\gamma' \cap P = \emptyset$. Therefore the projection of $(\gamma\cup\gamma')\backslash P$ on P is not connected, a contradiction.

Suppose now that γ' enters P in $x' \neq x$. Let $\overline{\gamma}$ (resp. $\overline{\gamma'}$) be the initial subgeodesic of γ (resp. γ') whose final point is x (resp. x'). The projection of $\overline{\gamma}$ on P is x and the projection of γ' on P is x'. But $\overline{\gamma} \cap \overline{\gamma'}$ contains p, and therefore their projections on P cannot be disjoint, a contradiction. One can proceed similarly for y, y', considering final subgeodesics instead of initial subgeodesics.

□

From now until the end of the section, fix a family $\{(P_i, r_i)\}_{i\in I}$ of homogeneous geodesic complete pointed metric spaces. Throughout the section all P-geodesics are implied to have range $\{(P_i, r_i)\}$.

If $\mathcal{I}$ is a family of subintervals of $[0, l]$ we set, for $x > 0$, $\mathcal{I}[x] = \{J \in \mathcal{I} | J \subseteq [0, x]\}$.

Definition 6.31. We will say that the P-geodesics Γ and Γ' with associated almost fillings, respectively, $\mathcal{I}_\Gamma$ and $\mathcal{I}'_\Gamma$ have the same P-pattern until $x > 0$ if

1. $\mathcal{I}_\Gamma[x] = \mathcal{I}_{\Gamma'}[x]$,
2. $\Gamma(I) = \Gamma'(I)\ \forall I \in \mathcal{I}_\Gamma[x]$,
3. if there exists $J \in \mathcal{I}_\Gamma$ such that $x \in J$ and x is not the first point of J, then there exists $J' \in \mathcal{I}_{\Gamma'}$ with the same property and $h_\Gamma(J) = h_{\Gamma'}(J')$.

We will say that Γ and Γ' have the same initial P-pattern if there exists some $x > 0$ such that Γ and Γ' have the same pattern until x.

Clearly, having the same initial P-pattern is an equivalence relation on the set of P-geodesics. Denote by $\mathcal{W}$ the quotient set.

The property of having the same initial P-pattern is modelled on the following property for geodesics.

Definition 6.32. Let γ, γ' be geodesics in $\mathbb{F}$ parametrized by arc length both starting from the same point p. We will say that γ and γ' have the same initial pattern if there exists $x > 0$ and a piece P such that $\gamma(x)$ and $\gamma'(x)$ both belong to P.

Lemma 6.33.

1. *Consider geodesics γ and γ' parametrized by arc length starting from p. If there exists a piece P such that, for some $x > 0$, $\gamma(x)$ and $\gamma'(x)$ both belong to P, then for each $0 \leq y \leq x$ there exists a piece P_y such that $\gamma(y), \gamma'(y) \in P_y$.*

2. *Having the same initial pattern is an equivalence relation.*

Proof. (1) If $p \in P$, the claim follows from the fact that each piece is convex. If this is not the case there exists x' such that $\gamma(x') = \gamma'(x') = \pi_P(p)$, and, for $x' \leq y \leq x$, $\gamma(y), \gamma'(y) \in P$. For $0 \leq y \leq x'$, and y contained in a non-trivial interval I such that $\gamma(I) \subseteq P'$ for some piece P', the claim follows from the proof of Lemma 6.29, point (3), which shows that $\gamma'(I) \subseteq P'$ as well. Also, if $I = [t_1, t_2]$ is maximal with that property, $\gamma(t_i) = \gamma'(t_i)$. If y is not contained in such an interval, then $\gamma(y) = \gamma'(y)$ because the union of maximal intervals as above is dense in $[0, x']$, and so y is the limit of a sequence of endpoints of such intervals.

(2) Consider geodesics parametrized by arc length $\gamma, \gamma', \gamma''$ and $x, y > 0$ such that $\gamma(x)$ and $\gamma'(x)$ (resp. $\gamma'(y)$ and $\gamma''(y)$) both belong to some piece P_1 (resp. P_2). By point (1), we can assume $y = x$. If $\gamma(x) = \gamma'(x)$ or $\gamma'(x) = \gamma''(x)$, we are done. Assuming that this is not the case, we will prove that $P_1 = P_2$. In fact, in this case it is easily seen that $\gamma'(x) \neq \pi_{P_1}(p), \pi_{P_2}(p)$, and therefore $\gamma'|_{[0,x]}$ contains non-trivial final subsegments contained in P_1 and P_2. So, $P_1 \cap P_2$ contains more than one point and $P_1 = P_2$, as required. □

The importance of this notion is due to the following lemma:

Lemma 6.34. *If γ and γ' are geodesic starting from p which have different initial patterns then $\gamma^{-1}\gamma'$ is a geodesic.*

Proof. It is clear that γ^{-1} and γ' concatenate well. □

Lemma 6.35. *If γ, γ' have the same initial pattern, then the induced P-geodesics Γ and Γ' have the same initial P-pattern.*

Proof. Let x and P be as in the definition of having the same initial pattern. If p is contained in P, then P contains the starting and ending point of the geodesics $\gamma|_{[0,x]}$, $\gamma'|_{[0,x]}$ and therefore they are contained in P. In this case $\mathcal{I}_\Gamma[x/2] = \mathcal{I}_\Gamma[x/2] = \emptyset$ and $h_\Gamma(J) = h_{\Gamma'}(J') = i$, where J, J' are maximal closed-open intervals such that $\gamma(J), \gamma'(J')$ are contained in P and i is chosen in such a way that P is isometric to P_i.

If $p \notin P$, then both γ and γ' must pass through the projection $\gamma(y)$ of p on P (and $y > 0$). It is easy to prove (see the proof of point (3) of Lemma 6.29) that Γ and Γ' have the same pattern until y.

□

Denote by $\mathcal{Y}(\mathbb{F}, p)$ the quotient of the set of geodesics starting from p by the equivalence relation of having the same initial pattern. The above lemma tells us that there is a well defined map $F_{\mathbb{F},p} : \mathcal{Y}(\mathbb{F}, p) \to \mathcal{W}$ (recall that $\mathcal{W}$ is the set of equivalence classes of P-geodesics with the same initial P-pattern).

6.4 Homeomorphism classes of asymptotic cones of relatively hyperbolic groups

Now we will analyze asymptotic cones of relatively hyperbolic groups. In each asymptotic cone X of a group G relatively hyperbolic with respect to its infinite subgroups of infinite index $H_1, \dots, H_n$, we fix the set of pieces to be the one containing the following:

- the subsets of X induced by a *lateral class of some $^*H_i$ which are not real trees (notice that if it is not empty this is a set of pieces for X, and if it is empty X is a real tree),
- if the collection $\mathcal{H}$ described above is a set of pieces, the transversal trees with respect to $\mathcal{H}$, and X otherwise.

The pieces as in the second point will be referred to, with an abuse, as transversal trees. If $\mathcal{H}$ is a set of pieces, by Proposition 6.21 they are homogeneous real trees of valency $2^{\aleph_0}$. On the other hand, if $\mathcal{H}$ is not a set of pieces, the valency of X is once again $2^{\aleph_0}$. In fact, the set P induced by any H_i is not a point, as each H_i is infinite, and P belongs to a set of pieces for X. Hence, being a homogeneous real tree, it contains a geodesic line. So, applying Lemma 6.20, we easily obtain that X has valency at least $2^{\aleph_0}$, and hence exactly $2^{\aleph_0}$.

Fix a group G, hyperbolic relative to $H_1, \dots, H_n$. Let $\mathcal{P} = \{(P_i, r_i)\} \cup \{(T, p_T)\}$ be representatives for the isometry classes of the pieces, where T is a homogeneous real tree with valency $2^{\aleph_0}$. We will denote by w_t the class in $\mathcal{W}$ of a P-geodesic Γ with associated almost filling of $[0, 1]$ simply $\{[0, 1)\}$ and such that $\Gamma(0) \in T$.

Definition 6.36. A choice of pair identifications is the choice of an identification of (P, p) with an element of $\mathcal{P}$, for each piece P and $p \in P$.

As we will see, the P-geodesics defined below are the ones represented by geodesics in asymptotic cones of relatively hyperbolic groups.

Definition 6.37. A P-geodesic Γ is admissible if for each $I_1 = [p_1, q_1), I_2 = [p_2, q_2)$ in its associated almost filling and such that $q_1 = p_2$, $\Gamma(p_1) \notin T$ or $\Gamma(p_2) \notin T$.

Proposition 6.38. *If $\mathbb{G}$ is an asymptotic cone of G, for each $p \in \mathbb{G}$ there exists a choice of pair identifications such that*

- $|F^{-1}_{\mathbb{G},p}(w_t)| = 1$,
- $|F^{-1}_{\mathbb{G},p}(w)| = 2^{\aleph_0}$ *if $w_t \neq w \in \mathcal{W}$ and w has an admissible representative,*
- $|F^{-1}_{\mathbb{G},p}(w)| = 0$ *otherwise.*

Proof. Suppose (without loss of generality) that $\mathbb{G} = C(G, e, \nu)$ and that $p = e$.

We want to express internally the property "the internal *geodesic $\gamma : [0, l\nu] \to {}^*\mathcal{CG}_S(G)$ induces in the asymptotic cone a geodesic contained in the piece P", for l finite. Let $J \subseteq \{1, \dots, n\}$ be the set of the indices j such that $C(H_j, e, \nu)$ is not a real tree. For $j \in J$, set $C_j = C(H_j, e, \nu)$ (here H_j is considered as endowed with the subspace metric inherited from G). We have 2 cases to consider, whether P is isometric to some C_j or it is a transversal tree.

In the first case, in order to express that property internally it is enough to find an infinitesimal η such that $d(\gamma(t), g\,{}^*H_j) \leq \eta\nu$ for each t in the domain of γ, where $g\,{}^*H_j$ induces P. Such an infinitesimal η exists because for each $r \in \mathbb{R}^+$ $d(\gamma, g\,{}^*H_i) \leq r\nu$ if the projection of γ is contained in P, and therefore we can use Lemma 2.14.

If P is isometric to T, the task is more difficult. Let M be as in Lemma 5.4. If the projection of γ is contained in a transversal tree, using the property of M and an argument based on Lemma 2.14, we get that there exists an infinitesimal η such that for each *lateral class H of some H_j ($j \in J$), the diameter of $\gamma \cap N_M(H)$ is bounded by $\eta\nu$. We claim that the converse holds as well.

In fact, consider γ such that there are two points $\gamma(x)$ and $\gamma(y)$, with $x < y$, such that $d(\gamma(x), \gamma(y)) \equiv \nu$, but there exists a *lateral class H of some H_j such that $d(x, H), d(y, H) \leq \eta'\nu$, for some infinitesimal η'. By the property of M, there are points $\gamma(x'), \gamma(y')$ whose distance from H is at most M. What is more, we can assume that $d(\gamma(x'), \gamma(y')) \equiv \nu$ by taking the x' as close as possible to x and y' as close as possible to y. This is a contradiction since any arc between $[x']$ and $[y']$ is contained in P. In particular this applies to a non-trivial subgeodesic of the geodesic δ induced by γ, and therefore δ is not contained in a transversal tree. This completes the proof of the claim.

Let us consider an admissible P-geodesic Γ, with associated almost filling of $[0,l]$ $\{I_a = [p_a, q_a)\}_{a \in A}$, and some finite $A' \subseteq A$. We want to find an internal non-empty set $\mathcal{G} = \mathcal{G}(A')$ of internal *geodesics $\hat{\gamma}$ such that their projections γ satisfy the properties required by Γ for $\{I_a\}_{a \in A'}$. Let us make this more precise. Choose for each i an identification of (P_i, r_i) with the appropriate $(C_{j(i)}, e)$. Also, choose a representative $u_a \in {}^*H_{j(i)}$ for each $\Gamma(p_a) \in P_i$. We require for the internal *geodesics $\hat{\gamma} \in \mathcal{G}$ to satisfy the following, for each $a \in A'$

1. $\hat{\gamma}(0) = e$,
2. $\gamma|_{I_a}$ is contained in a piece P isometric to the only $Q \in \mathcal{P}$ containing $\Gamma(p_a)$,
3. I_a is maximal with property (2),
4. if P is induced by a *lateral class of $^*H_j$ ($j \in J$), there exist an infinitesimal $\eta = \eta(A')$ and $g \in G$ such that $d(g, \hat{\gamma}(p_a\nu)), d(\hat{\gamma}(q_a\nu), gu_a) \leq \eta\nu$.

Suppose that we are able to prove that there actually exist internal geodesics with these properties, for each finite $A' \subseteq A$, as we will do later. Then we have that, for each A' and fixing a sufficiently large infinitesimal $\rho(A')$, we can express the 4 properties internally (see the first part of the proof). Therefore, after choosing an infinitesimal greater than any $\rho(A')$, we can use $\aleph_0-$ saturation to find an internal *geodesic $\hat{\gamma}$ which satisfies $(1)-(4)$ for each $a \in A$. Let γ be its projection. It is quite clear that we can choose identifications of each pair (P, p) with some (P_i, r_i) or (T, p_T), where P is a piece intersecting γ in a non-trivial subgeodesic and p is the entrance point of γ in P, in such a way that the P-geodesic associated to γ is Γ. In fact, in the case that (P, p) has to be identified with some (P_i, r_i), we can use g as in property (4) to "translate" the fixed identification of (P_i, r_i) with $(C_{j(i)}, e)$. In the case that (P, p) has to be identified with (T, p_T), we can use the "isotropy" of T, that is the fact that for each $x, y \in T$ with $d(x, p_T) = d(y, p_T)$ there exists an isometry of T fixing p_T and taking x to y.

So far we proved that $|F^{-1}_{\mathbb{G},e}([\Gamma])| \geq 1$ (for an admissible Γ). We want to use the internal geodesic γ we found above to construct many other geodesics with the same properties. Consider a hyperbolic element of infinite order $g \in G$. Notice that the isometry induced in $\mathbb{G}$ by left multiplication by g stabilizes no piece which is not a transversal tree. This immediately implies that, unless an initial subgeodesic of γ is contained in the transversal tree at e, γ and $g\gamma$ do not have the same initial pattern. Similarly, if $n_1 \neq n_2 \in {}^*\mathbb{N}$

(and $d(e,g^{n_i})\in o(\nu)$), $g^{n_1}\gamma$ and $g^{n_2}\gamma$ do not have the same initial pattern. Notice that the cardinality of $\{n\in {}^*\mathbb{N} : d(e,g^n)\in o(\nu)\}$ is at least $2^{\aleph_0}$. Therefore, if $[\Gamma]\neq w_t$ and Γ is admissible, we have $|F^{-1}_{\mathbb{G},e}(w)|\geq 2^{\aleph_0}$. The other inequality clearly holds, so we are done in this case. On the other hand, it is clear from the definitions that $|F^{-1}_{\mathbb{G},e}(w_t)|=1$, and that no geodesic in $\mathbb{G}$ has non-admissible associated P-geodesic, that is $|F^{-1}_{\mathbb{G},e}(w)|=0$ if w has no admissible representatives.

We are only left with finding internal geodesics as above. For each $a\in A'$ we can find a geodesic parametrized by (a translate of the) arc length $\gamma_a:[p_a,q_a]\to\mathbb{G}$ which is contained in a piece isometric to $P_{h_\Gamma(a)}$, $\gamma_a(p_a)=e$ and $\gamma_a(q_a)=[u_a]$. Order A' in such a way that $a\leq b$ if $p_a\leq p_b$. We want to show that if $a<b$, up to translating γ_b by an element of ${}^*G$, we can find a geodesic parametrized by a translate of the arc length $\gamma:[p_a,q_b]\to\mathbb{G}$ such that $\gamma|_{[p_a,q_a]}=\gamma_a$ and $\gamma_{[p_b,q_b]}=\gamma_b$. In fact, suppose first that $q_a<p_b$. We can find a geodesic δ of length q_b-q_a starting from $\gamma_a(q_a)$ such that γ_a and δ concatenate well. Also, there exists an element $g\in {}^*G$ such that $g\gamma_b$ has starting point the final point of δ. Up to changing g we can also arrange that δ and γ_b concatenate well. The concatenation of γ_a, δ and $g\gamma_b$ is the required geodesic. If $q_a=p_b$, we can still find g such that γ_a and γ_b concatenate well, unless they are both contained in a transversal tree, but this is not the case as Γ is admissible.

Using inductively the argument above (and, possibly, the first part of it for the minimum and maximum of A'), we obtain a geodesic γ such that $\gamma|_{I_a}$ is contained in a piece isometric to $P_{h_\Gamma(a)}$ for each $a\in A'$. An internal *geodesic connecting e to an element of ${}^*G$ which projects on the last point of γ satisfies all our requirements.

□

Definition 6.39. Consider groups G_i, for $i=0,1$, which are hyperbolic relative to $\mathcal{H}_i=\{H_i^1,\dots,H_i^{n(i)}\}$, and $\nu\gg 1$. We will say that G_0 and G_1 are comparable at scale ν if for each $H\in\mathcal{H}_i$ such that $C(H,e,\nu)$ is not a real tree there exists $H'\in\mathcal{H}_{i+1}$ such that $C(H',e,\nu)$ is bilipschitz equivalent to $C(H,e,\nu)$.

Lemma 6.40. *Consider relatively hyperbolic groups G_0, G_1 which are comparable at scale ν and let $C(G_i)$ be the asymptotic cone of G_i with scaling factor ν. There is a metric d' on $C(G_1)$, bilipschitz equivalent to its original one, such that each piece of $C(G_1)$ is isometric to a piece of $C(G_2)$. Also, the conclusion of Proposition 6.38 still holds for $\mathbb{G}=(C(G_1),d')$.*

Remark 6.41. It should be pretty clear to the the reader that $(C(G_1),d')$ is indeed tree-graded with respect to the same set of pieces as $(C(G_1),d)$.

Proof. Let $\mathcal{P}_i$ be the set of pieces for $C(G_i)$. For each $P\in\mathcal{P}_1$ choose a $k-$bilipschitz equivalence with an appropriate element of $\mathcal{P}_0$ (for k large enough this can be done, also using the fact that the valency of homogeneous trees determine their isometry type, see [DP]). There exists a metric d'_P on P such that this is an isometry. We want to prove that there is a metric d' on $C(G_1)$, bilipschitz equivalent to its original metric d, such that its restriction to each $P \in \mathcal{P}_1$ is d'_P. In order to do so, we will first assign a new length to each path whose length with respect to d is finite. This can be done as follows. Consider a path as above and parametrize it by arc length with respect to d to obtain $\gamma : [0,l] \to C(G_1)$. Let $\{I_a = [p_a, q_a]\}_{a\in A}$ be the set of maximal closed intervals I_a in $[0,l]$ such that $\gamma|_{I_a}$ is contained in one piece, denoted P_a. Let $X \subseteq [0,l]$ be the union of such intervals. We define

$$l'(\gamma) = \sum_{a\in A} d'_{P_a}(\gamma(p_a), \gamma(q_a)) + \lambda\left([0,l]\backslash X\right),$$

where λ is the Lebesgue measure. Notice that $\sum_{a\in A} d(\gamma(p_a),\gamma(q_a)) + \lambda\left([0,l]\backslash X\right) = \lambda(X)$, as γ is parametrized by arc length and the set of the endpoints of the intervals $[p_a, q_a]$ is easily seen to have null measure. It is clear now that $1/kl(\gamma) \le l'(\gamma) \le kl(\gamma)$. We can define

$$d'(x,y) = \inf_{\gamma\in\Gamma(x,y)} l'(\gamma),$$

where $\Gamma(x,y)$ is the set of paths with finite length with respect to d which connect x to y. We have that d' is bilipschitz equivalent to d and that it restricts to d'_P on each $P \in \mathcal{P}_1$. Also, $(C(G_1), d')$ is tree-graded with respect to $\mathcal{P}_1$.

It is easy to check that Proposition 6.38 still holds for $(C(G_1), d')$.

□

Theorem 6.42. *Suppose that G_0 and G_1 are relatively hyperbolic groups which are comparable at scale ν and let $C(G_i)$ be the asymptotic cone of G_i with scaling factor ν. Then $C(G_0)$ is bilipschitz homeomorphic to $C(G_1)$.*

Proof. Let us set $C(G_1) = \mathbb{F}$ and $C(G_2) = \mathbb{G}$. Consider $\mathbb{G}$ equipped with the metric d' as in the previous lemma. We want to show that $\mathbb{F}$ and $\mathbb{G}$ are isometric.

If $X \subseteq \mathbb{F}$ and $x \in X$ denote by $\mathcal{Y}(X,x)$ the set of elements of $\mathcal{Y}(\mathbb{F},x)$ which can be represented by a geodesic contained in X. We will call a subspace X of $\mathbb{F}$ *good* if it has the following properties:

1. X is geodesic.

2. For each $x \in X$, the set $\mathcal{Y}(X,x)$ either has at most 2 elements or it coincides with $\mathcal{Y}(\mathbb{F},x)$. In the first case x will be called empty for X, while in the second case it will be called full for X.

3. If X contains a non-trivial geodesic contained in one piece (or, equivalently, if it contains 2 points on the same piece), then it contains the entire piece.

Analogous definitions can be given for $\mathbb{G}$. Notice that an increasing union of good subspaces is a good subspace. Also, remark that if X is a good subspace of $\mathbb{F}$ or $\mathbb{G}$ and $x, y \in X$, then *any* geodesic between x and y is contained in X (i.e, X is convex). In fact, if γ, γ' are geodesics connecting x and y and $p \in \gamma \backslash \gamma'$, there exists a piece containing p and intersecting both γ and γ' in a non-trivial arc (as p is contained in a simple loop which is a union of two subgeodesics of γ and γ'). Therefore, conditions (1) and (3) imply the claim.

We wish to construct the required isometry using Zorn's Lemma on the set of *good pairs*, that is pairs (X, f) such that

- X is a good subspace of $\mathbb{F}$,
- f is an isometric embedding of X into $\mathbb{G}$ which preserves fullness, that is $f(x)$ is full for $f(X)$ whenever x is full for X,
- if, for some piece $P \subseteq \mathbb{G}$, $f(X) \cap P$ contains at least 2 points, there exists a piece P' of $\mathbb{F}$ such that $f(P') = P$.

Notice that if (X, f) is a good pair, $f(X)$ is a good subspace of $\mathbb{G}$ (we require the third property in order to have this). A point is a good subspace, therefore the set such pairs is not empty. If we set $(X, f) \leq (Y, g)$ when $X \subseteq Y$ and $g|_X = f$, then clearly any chain has an upper bound. Therefore there exists a maximal element (M, h). We want to show that $M = \mathbb{F}$. Notice that M is closed, because h can be extended to $\overline{M}$ as $\mathbb{G}$ is complete, and, as we are going to show, $\overline{M}$ is a good subspace.

Let us prove that $\overline{M}$ satisfies (3) first, as it is the simplest condition to check. If $[x, y]$ is a non-trivial geodesic contained in a piece P and x', y' are sufficiently close to x and y respectively, then any geodesic from x' to y' contains a non-trivial subgeodesic contained in P. This readily implies (3).

Let us prove (1). Consider any $x, y \in \overline{M}$. We want to show that there is a geodesic contained in $\overline{M}$ which connects them. Consider any geodesic γ in $\mathbb{F}$ from x to y. Consider any piece Q which intersects γ in a non-trivial arc between $\pi_Q(x) = x'$ and $\pi_Q(y) = y'$. Each geodesic between points close enough to x and y intersects Q in a non-trivial arc, and this readily implies, by conditions (1) and (3) for M, that $Q \subseteq M$. This argument shows that there exists a dense subset of γ contained in M (see Lemma 6.26). By the remark that each geodesic connecting two points in M is contained in M, we have that $\gamma \backslash \{x, y\} \subseteq M$, and therefore $\gamma \subseteq \overline{M}$.

We are left to show (2). First, we have that $\mathcal{Y}(M, x) = \mathcal{Y}(\overline{M}, x)$ if $x \in M$. In fact, if γ represents an element of $\mathcal{Y}(\overline{M}, x)$, by the previous argument $\gamma \backslash \{y\}$ is contained in M, where y is the last point of γ. An initial subgeodesic γ' of γ is contained in M and so $[\gamma] = [\gamma'] \in \mathcal{Y}(M, x)$. Also, $\mathcal{Y}(\overline{M}, x)$ cannot contain more than one element if $x \in \overline{M}\backslash M$. Suppose in fact that this is not the case and consider geodesics $\gamma_1, \gamma_2 \subseteq \overline{M}$ such that $[\gamma_1] \neq [\gamma_2] \in \mathcal{Y}(\overline{M}, x)$. By Lemma 6.34, the concatenation γ of γ_2^{-1} and γ_1 is a geodesic. By the proof of point (1), we would have $x \in M$, as it is not an endpoint of γ.

We have thus proved that $\overline{M}$ is good, so $M = \overline{M}$ by maximality and M is closed.

Assume that there exists $x \notin M$. Consider some $p' \in M$ and let p be the last point on a geodesic $[p', x]$ which lies on M. We want to show that p is empty for M, by showing that $[p, x]$ is not a representative of an element in $\mathcal{Y}(M, p)$. In fact, suppose that this is not the case. Then there exists a point $q \neq p$ on $[p, x]$, a piece P and a point $r \in P$, $r \neq p$, such that a geodesic $[p, r]$ is contained in M and $q, r \in P$. If $p \in P$, by property (3) we have $P \subseteq M$ and in particular $q \in M$, which contradicts our choice of p. If $p \notin P$, we can assume $r = \pi_P(p)$. So, $[p, q]$ must contain $r \in M$. This is a contradiction as $[p, q] \cap M = \{p\}$.

Fix a set of representatives R_1 (resp. R_2) for the elements of $\mathcal{Y}(\mathbb{F}, p)\backslash\mathcal{Y}(M, p)$ (resp. $\mathcal{Y}(\mathbb{G}, p)\backslash\mathcal{Y}(f(M), f(p))$). At first, we want to extend h to the union M' of M and all the elements of R_1. A consequence of Proposition 6.38 and of the last part of the statement of Lemma 6.40 is that, up to changing representatives of R_i, there is a bijection $b : R_1 \to R_2$ such that, for each $\gamma \in R_1$,

1. $b(\gamma)$ and γ have the same length,
2. $b(\gamma)$ and γ have the same associated P-geodesics (for some choices of pairs identifications).

In fact, given the choices of pairs identification as in Proposition 6.38 for p and $f(p)$, we clearly have $|F^{-1}_{\mathbb{F},p}(w)\backslash\mathcal{Y}(M, p)| = |F^{-1}_{\mathbb{F},p}(w)| = |F^{-1}_{\mathbb{G},f(p)}(w)| = |F^{-1}_{\mathbb{G},f(p)}(w)\backslash\mathcal{Y}(f(M), f(p))|$ for each $w_t \neq w \in \mathcal{W}$. But also in the case $w = w_t$ this holds, as a geodesic contained in a piece isometric to T is mapped by the isometry f to a geodesic with the same property. The above considerations show that (2) can be arranged. To obtain (1), it is enough to substitute geodesics in R_1 and R_2 with appropriate subgeodesics. We can define an extension of h, denoted $\overline{h}$, as follows:

$$\overline{h}(x) = \begin{cases} h(x) & \text{if} \quad x \in M \\ b(\gamma)(t) & \text{if} \quad x = \gamma(t) \text{ for some } \gamma \in R_1 \end{cases}$$

Notice that $\overline{h}$ is indeed an isometric embedding (see Lemma 6.34). The last step is to extend it further so that the domain satisfies property (3). Consider a piece P which intersects some $\gamma \in R_1$ in a non-trivial subgeodesic γ'. As $b(\gamma)$ and γ have the same associated P-geodesic, $\overline{h}(\gamma')$ is contained in a piece P' isometric to P. Not only that: using the fixed choices of pairs identifications, we have that there exists an isometry $f_P : P \to P'$ which maps γ' to $\overline{h}(\gamma')$. Let Δ be the family of pieces P as above. Consider the isometries $\{f_P\}_{P\in\Delta}$. We can use them to further extend $\overline{h}$ to $\widetilde{h} : M'' \to \mathbb{G}$, where $M'' = M' \cup \bigcup_{P\in\Delta} P$, as follows:

$$\widetilde{h}(x) = \begin{cases} h(x) & \text{if} \quad x \in M' \\ f_P(x) & \text{if} \quad x \in P \end{cases}$$

Once again, this is an isometric embedding. It is quite clear that M'' satisfies (1) and (3). It is also not difficult to see that it satisfies (2) as well, and more precisely that

- p is full for M'',
- each point in $M\backslash\{p\}$ is empty (resp. full) for M'' if and only if it is empty (resp. full) for M,
- each point in $M''\backslash M$ is empty for M''.

Also, h is readily checked to satisfy all requirements needed to establish that (M'', h) is a good pair. By maximality of M, this is a contradiction.

We finally proved that if (M, h) is a maximal good pair, then $M = \mathbb{F}$. Therefore, there exists an isometric embedding $h : \mathbb{F} \to \mathbb{G}$, with the further property that h preserves fullness. Let us show that this implies that h is surjective. Consider, by contradiction, some $x \in \mathbb{G}\backslash h(\mathbb{F})$. Fix some $p \in \mathbb{F}$ and consider a geodesic $[h(p), x]$. As $h(\mathbb{F})$ is closed, being a complete metric space, we can assume that $[h(p), x] \cap h(\mathbb{F}) = \{h(p)\}$. Repeating an argument we already used for M (recall that $h(\mathbb{F})$ is a good subspace), we have that $[h(p), x]$ represents an element of $\mathcal{Y}(\mathbb{G}, h(p))\backslash\mathcal{Y}(h(\mathbb{F}), h(p))$. But p is full for $\mathbb{F}$, so this contradicts the hypothesis that h preserves fullness.

$\square$

Chapter 7

Universal tree-graded spaces

Fix, throughout the chapter, a family $\{(P_i, r_i)\}$ of complete homogeneous geodesic pointed metric spaces. We also assume that P_i is not isometric to P_j if $i \neq j$ and that no P_i consists of a single point.

Recall that $\mathcal{W}$ is the set of equivalence classes of P-geodesics with the same initial P-pattern. In view of the results in the previous chapter, it is a natural problem to find those assignments $(w \in \mathcal{W}) \mapsto \alpha(w)$, where each $\alpha(w)$ is a cardinality, which are realized by a homogeneous tree-graded space, and to determine if this assignment characterizes tree-graded spaces. In this chapter, we will answer those questions in the particular case that each $\alpha(w)$ is infinite.

Our aim is now to construct a "universal" tree-graded space, given an infinite cardinality $\alpha(w)$ for each $w \in \mathcal{W}$.

Theorem 7.1. *Consider any assignment $(w \in \mathcal{W}) \mapsto \alpha(w)$, where each $\alpha(w)$ is infinite. There exists a tree-graded space $\mathbb{F} = \mathbb{F}_\alpha$ with trivial transversal trees such that each of its pieces is isometric to one of the P_i's and, for an appropriate choice of pair identifications, for each $w \in \mathcal{W}$ and $p \in \mathbb{F}$, $F^{-1}_{\mathbb{F},p}(w)$ has cardinality $\alpha(w)$.*

Remark 7.2. If we did not require the $\alpha(w)$'s to be infinite the theorem would be false, for if the cardinality of some $F^{-1}_{\mathbb{F},p}(w)$ is a most one, many other cardinalities are forced to be 0 (for the same reason why only admissible P-geodesics are represented by geodesics in the asymptotic cone of a relatively hyperbolic group). It seems reasonable that the theorem can be extended (in the same generality) to the case that the $\alpha(w)$'s are at least 2.

It makes sense to call $\mathbb{F}$ as above "universal" tree-graded space in view of Theorem 7.6, at the end of the chapter.

Proof. Denote by W the set of all P-geodesics. For $\Gamma \in W$, we will denote by $[\Gamma]$ the corresponding class in $\mathcal{W}$. Let us define $\mathbb{F}$, at first as a set. Some of the definitions which follow are inspired by the definitions of A_μ and of its distance in [DP]. Set $\alpha = \sup_{w\in\mathcal{W}} \alpha(w)$. We will need to fix for each i and $x \in P_i$ different from r_i an isometry ϕ_x of P_i which maps x to r_i. If Γ is a P-geodesic with associated almost filling of $[0,l]$ $\mathcal{I} = \{[p_a, q_a)\}$, and $x < y \in [0,l]$ do not lie in the internal part of any $I \in \mathcal{I}$, denote by

- $-\Gamma$ the P-geodesic with associated almost filling (once again of $[0,l]$) $\{[l-q_a, l-p_a)\}$ and such that $-\Gamma(l-q_a) = \phi_{\Gamma(p_a)}(r_{h_\Gamma(p_a)})$ (where h_Γ denotes as usual the index selector of Γ),
- $\Gamma^{x,y}$ the P-geodesic with associated almost filling (of $[0, y-x]$) $\{[p_a - x, q_a - x) : p_a \geq x, q_a \leq y\}$ and such that $\Gamma^{x,y}(t) = \Gamma(t+x)$.

The idea is that $-\Gamma$ moves backwards along Γ, and $\Gamma^{x,y}$ is a restriction of Γ.

The elements of $\mathbb{F}$ will be quadruples $f = (\rho_f, \Gamma_f, \mathcal{I}_f, \beta_f)$ such that

1. $\rho_f \in \mathbb{R}_{\geq 0}$,
2. $\mathcal{I}_f$ is an almost filling of $[0, \rho_f]$,
3. Γ_f is a P-geodesic with associated almost filling $\mathcal{I}_f$,
4. $\beta_f : [0, \rho_f) \to \alpha$ is piecewise constant from the right, that is for each t there exists $\epsilon > 0$ such that $f|_{[t,t+\epsilon]}$ is constant,
5. if x lies in the internal part of some $I \in \mathcal{I}_f$, β_f is constant in a neighborhood of x,
6. $\beta_f(t) < \alpha([\Gamma_f^{t,\rho_f}])$ for each $t \in [0, \rho_f)$ such that $t = 0$ or β_f is not constant in a neighborhood of t.

Let us construct some examples of elements of $\mathbb{F}$. If $x \in P_i$ and $\mu < \alpha$, denote by $f^{x,\mu}$, if it exists, the element of $\mathbb{F}$ such that

- $\rho_{f^{x,\mu}} = d_{P_i}(r_i, x)$,
- $\mathcal{I}_{f^{x,\mu}} = \{[0, \rho_{f^{x,\mu}})\}$,
- $\Gamma_{f^{x,\mu}}(0) = x$,
- $\beta_{f^{x,\mu}}$ is constantly μ,

Condition (6) restricts the possible values of μ.

We are now going to define a concatenation of elements of $\mathbb{F}$. Consider $f, g \in \mathbb{F}$. The concatenation $f * g$ is the element of $\mathbb{F}$ such that

- $\rho_{f*g} = \rho_f + \rho_g$,

- $\mathcal{I}_{f*g} = \mathcal{I}_f \cup \{\rho_f + I : I \in \mathcal{I}_g\}$,
- $\Gamma_{f*g}^{0,\rho_f} = \Gamma_f$ and $\Gamma_{f*g}^{\rho_f,\rho_f+\rho_g} = \Gamma_g$
- $\beta_{f*g}(t) = \beta_f(t)$, where $\beta_f(t)$ is defined and $\beta_{f*g}(t) = \beta_g(t-\rho_f)$ where $\beta_g(t-\rho_f)$ is defined,

We want now to define a distance on $\mathbb{F}$. Consider $f, g \in \mathbb{F}$. Let $s = s(f,g)$ be their separation moment, i.e.

$$s = \sup\{t|\forall t' \in [0,t]\ \Gamma_f(t') = \Gamma_g(t'), \beta_f(t') = \beta_g(t')\}.$$

Notice that this supremum is never a maximum. We will consider 2 cases.

- (a) If $\beta_f(s) = \beta_g(s)$ and $h_{\Gamma_f}(s) = h_{\Gamma_g}(s) = i$ (in particular they are defined in s), denoting by $J_f \in \mathcal{I}_f$ and $J_g \in \mathcal{I}_g$ the intervals containing s, we set

$$d(f,g) = (\rho_f - s) + (\rho_g - s) + d_{P_i}(\Gamma_f(s), \Gamma_g(s)) - l(J_f) - l(J_g),$$

- (b) in any other case

$$d(f,g) = (\rho_f - s) + (\rho_g - s).$$

For later purposes, define $u = u(f,g)$ and $v = v(f,g)$ in the following way:

- if $d(f,g)$ is as in case (a), let u and v be such that $J_f = [s,u)$, $J_g = [s,v)$.
- if $d(f,g)$ is as in case (b), set $u = v = s$,

The following remark will be used many times in the proof that d is a distance.

Remark 7.3.

- s does not lie in the internal part of any element of $\mathcal{I}_f$ or $\mathcal{I}_g$,
- if $u > s$ or $v > s$, then both inequalities hold and $[s,u) \in \mathcal{I}_f$, $[s,v) \in \mathcal{I}_g$,
- $s \le u \le \rho_f$, $s \le v \le \rho_g$,
- the formula in case (a) can be rewritten as $d(f,g) = (\rho_f - u) + (\rho_g - v) + d_{P_i}(\Gamma_f(s), \Gamma_g(s))$,
- $(\rho_f - u) + (\rho_g - v) \le d(f,g) \le (\rho_f - s) + (\rho_g - s)$,
- if $s(f,g) < s(g,h)$, then $s(f,h) = s(f,g)$.

Lemma 7.4. *d is a distance.*

Proof. The only non trivial property to check is the triangular inequality. Consider $f, g, h \in \mathbb{F}$. We have to show that $d(f,h) \leq d(f,g) + d(g,h)$. Set $s_1 = s(f,g)$, $s_2 = s(g,h)$ and $s_3 = s(f,h)$. Define analogously u_i and v_i, $i = 1,2,3$. We will consider several cases, which cover all possible situation up to exchanging the roles of f and h (and therefore, for example, u_1 and v_2).

1) $u_1 \leq s_3$, $v_2 \leq s_3$. In this case we get

$$d(f,h) \leq (\rho_f - s_3) + (\rho_h - s_3) \leq (\rho_f - u_1) + (\rho_h - v_2) \leq$$

$$(\rho_f - u_1) + (\rho_g - v_1) + (\rho_g - u_2) + (\rho_h - v_2) \leq d(f,g) + d(g,h).$$

2) $s_3 < u_1 \leq u_3$, $s_1 < u_1$ and $v_2 \leq v_3$. We have that $[s_3, u_3)$ and $[s_1, u_1)$ both belong to $\mathcal{I}_f$ and their intersection in not empty. Therefore $[s_3, u_3) = [s_1, u_1)$, that is, $s_3 = s_1$ and $u_3 = u_1$. Also, clearly $s_2 \geq s_3$, by the definition of separation moment. We will consider 2 subcases.

2$'$) $s_2 < v_2$. In this case, by the same argument we just used, $s_2 = s_3 = s_1$ and $v_2 = v_3$. For $i = h_{\Gamma_f}(s_3)$ (we will not repeat this), using the relations we found so far and the triangular inequality in P_i, we have that

$$d(f,h) = (\rho_f - u_3) + d_{P_i}(\Gamma_f(s_3), \Gamma_h(s_3)) + (\rho_h - v_3) \leq$$

$$(\rho_f - u_1) + d_{P_i}(\Gamma_f(s_1), \Gamma_g(s_1)) + (\rho_g - v_1) + (\rho_g - v_2) + d(\Gamma_g(s_2), \Gamma_h(s_2)) + (\rho_h - v_2)$$
$$= d(f,g) + d(g,h).$$

If $s_2 = v_2$, we have $s_2 \in [s_3, v_3]$. But s_2 cannot belong to the internal part of $[s_3, v_3) \in \mathcal{I}_h$. Therefore either $s_2 = s_3$ or $s_2 = v_3$. But $s_2 = s_3$ is contradictory as it implies $\beta_g(s_2) = \beta_g(s_1) = \beta_f(s_1) = \beta_f(s_3) = \beta_h(s_3) = \beta_h(s_2)$ and similarly $h_{\Gamma_g}(s_2) = h_{\Gamma_h}(s_2)$, therefore we should have $s_2 < v_2$.

2$''$) $v_2 = s_2 = v_3$. As $[s_1, v_1) \in \mathcal{I}_g$, $[s_1, v_2) = [s_3, v_3) \in \mathcal{I}_h$ and v_3 is the separation moment of g and h, we get $v_1 = v_2$. We have, using $s_1 = s_3 < s_2$ (and the definition of separation moment),

$$d(f,h) = (\rho_f - u_3) + d_{P_i}(\Gamma_f(s_3), \Gamma_h(s_3)) + (\rho_h - v_3) =$$

$$(\rho_f - u_1) + d_{P_i}(\Gamma_f(s_1), \Gamma_g(s_1)) + (\rho_h - v_2) \leq$$

$$(\rho_f - u_1) + d_{P_i}(\Gamma_f(s_1), \Gamma_g(s_1)) + (\rho_g - v_1) + (\rho_h - u_2) + (\rho_h - v_2) \leq d(f,g) + d(g,h).$$

3) $s_3 < u_1 \leq u_3$, $s_1 = u_1$ and $v_2 \leq v_3$. As s_1 cannot lie in the internal part of $[s_3, u_3) \in \mathcal{I}_f$, $s_1 = u_1 = u_3$. Up to exchanging the roles of f and h we already treated the case that $s_2 < v_2$ (case 2$''$). So, we can assume $s_2 = v_2$. As $s_1 = u_3 > s_3$, we have $s_2 = s_3$, in particular $s_1 > s_2$. But $\beta_g(s_2) = \beta_f(s_2) = \beta_f(s_3) = \beta_h(s_3) = \beta_h(s_2)$ and analogously $h_{\Gamma_g}(s_2) = h_{\Gamma_h}(s_2)$, so we should have $s_2 < v_2$, a contradiction. This (sub)case is therefore impossible.

4) $u_1 = u_3 = s_3$, $v_2 \le v_3$. Notice that $v_3 = s_3$, and so $v_2 \le s_3$.

$$d(f,h) = (\rho_f - s_3) + (\rho_h - s_3) \le (\rho_f - u_1) + (\rho_g - v_1) + (\rho_g - u_2) + (\rho_h - v_2) \le$$

$$d(f,g) + d(g,h).$$

5) $u_1 > u_3 > s_3$. In this case we have $s_1 \ge u_3$ (if $s_1 = u_1$ it is obvious, if $s_1 < u_1$ it follows from the fact that u_3 cannot lie in the internal part of $[s_1, u_1))$. Also, $s_2 = \min\{s_3, s_1\} = s_3$. Observe that $s_2 < u_2$, as $\beta_g(s_2) = \beta_f(s_2) = \beta_f(s_3) = \beta_h(s_3) = \beta_h(s_2)$ and similarly $h_{\Gamma_g}(s_2) = h_{\Gamma_h}(s_2)$ (we used $s_1 \ge u_3 > s_3 = s_2$, $s_3 = s_2$ and $u_3 > s_3$). Notice that $v_2 = v_3$ and $u_2 = u_3$. In fact, $[s_2, v_3) = [s_3, v_3) \in \mathcal{I}_h$ and $[s_2, u_3) = [s_3, u_3) \in \mathcal{I}_f$, but also $[s_3, u_3) \in \mathcal{I}_g$ as $u_3 < u_1$. If $s_1 = u_1$, we have

$$d(f,h) = (\rho_f - u_3) + d_{P_i}(\Gamma_f(s_3), \Gamma_h(s_3)) + (\rho_h - v_3) \le$$

$$(\rho_f - s_1) + (s_1 - u_3) + 2(\rho_g - s_1) + d_{P_i}(\Gamma_g(s_2), \Gamma_h(s_2)) + (\rho_h - v_2) =$$

$$(\rho_f - s_1) + (\rho_g - s_1) + (\rho_g - u_2) + d_{P_i}(\Gamma_g(s_2), \Gamma_h(s_2)) + (\rho_h - v_2) = d(f,g) + d(g,h).$$

If $s_1 < u_1$ and $j = h_{\Gamma_f}(s_1)$, the chain of inequalities can be modified as follows:

$$d(f,h) = (\rho_f - u_3) + d_{P_i}(\Gamma_f(s_3), \Gamma_h(s_3)) + (\rho_h - v_3) \le$$

$$(\rho_f - u_1) + (u_1 - s_1) + (s_1 - u_3) + 2(\rho_g - v_1) + [(v_1 - s_1) - (v_1 - s_1)] +$$

$$d_{P_i}(\Gamma_g(s_2), \Gamma_h(s_2)) + (\rho_h - v_2) =$$

$$(\rho_f - u_1) + d_{P_j}(\Gamma_f(s_1), r_j) + (\rho_g - v_1) + (\rho_g - u_2) - (v_1 - s_1) +$$

$$d_{P_i}(\Gamma_g(s_2), \Gamma_h(s_2)) + (\rho_h - v_2) \le$$

$$(\rho_f - u_1) + d_{P_j}(\Gamma_f(s_1), \Gamma_g(s_1)) + (v_1 - s_1) - (v_1 - s_1) + (\rho_g - v_1) + d(g,h) =$$

$$d(f,g) + d(g,h).$$

6) $u_1 > u_3 = s_3$. As in case 5), $s_1 \ge u_3$, so $s_1 \ge s_3$. Notice that $s_2 \ge s_3$. If $s_1 = u_1$, we also have $s_1 > s_3$ and hence $s_2 = s_3$. Also, $\beta_g(s_2) = \beta_f(s_2) = \beta_f(s_3) \ne \beta_h(s_3) = \beta_h(s_2)$, hence $u_2 = v_2 = s_2$ (we used $s_1 > s_3 = s_2$, $s_3 = u_3$ and $s_3 = s_2$).

$$d(f,h) = (\rho_f - s_3) + (\rho_h - s_3) \le (\rho_f - s_1) + (s_1 - s_3) + 2(\rho_g - s_1) + (\rho_h - s_2) =$$

$$(\rho_f - s_1) + (\rho_g - s_1) + (\rho_g - s_2) + (\rho_h - s_2) = d(f,g) + d(g,h).$$

We are left to deal with the case $s_1 < u_1$, which has 2 subcases

6′) $s_1 = s_3$. In this case $s_2 = s_3$, for otherwise (i.e. for $s_2 > s_3 = s_1$) we would have $\beta_f(s_3) = \beta_f(s_1) = \beta_g(s_1) = \beta_h(s_1) = \beta_h(s_3)$ and similarly $h_{\Gamma_f}(s_3) = h_{\Gamma_h}(s_3)$, so $s_3 < u_3$ (we used $s_3 = s_1$, $s_1 < u_1$, $s_1 < s_2$ and $s_1 =$

s_3). Also, $s_2 = u_2$ as $\beta_g(s_2) = \beta_g(s_1) = \beta_f(s_1) = \beta_f(s_3) \neq \beta_h(s_3) = \beta_h(s_2)$ (we used $s_2 = s_1$, $s_1 < u_1$, $s_1 = s_3$, $s_3 = u_3$ and $s_3 = s_2$).

6″) $s_1 > s_3$. Also in this case $s_2 = s_3$, and $s_2 = u_2$ because $\beta_g(s_2) = \beta_g(s_3) = \beta_f(s_3) \neq \beta_h(s_3) = \beta_h(s_2)$.

In both cases 6′) and 6″) the following estimate holds:

$$d(f,h) = (\rho_f - s_3) + (\rho_h - s_3) \leq (\rho_f - u_1) + d_{P_i}(\Gamma_f(s_1), r_i) + (s_1 - s_3) +$$

$$2(\rho_g - v_1) + (\rho_h - s_2) \leq$$

$$(\rho_f - u_1) + d_{P_i}(\Gamma_f(s_1), \Gamma_g(s_1)) + (\rho_g - v_1) + (v_1 - s_1) + (s_1 - s_3) + (\rho_g - v_1) + (\rho_h - s_2)$$

$$(\rho_f - u_1) + d_{P_i}(\Gamma_f(s_1), \Gamma_g(s_1)) + (\rho_g - v_1) + (\rho_g - s_2) + (\rho_h - v_2) = d(f,g) + d(g,h).$$

□

Lemma 7.5. $\mathbb{F}$ *is complete.*

Proof. Notice that $d(f,g) \geq |\rho_f - \rho_g|$. Therefore, given a Cauchy sequence f_n we have that $\rho_{f_n} \to \rho$, for some $\rho \geq 0$. If for some $t \in [0,\rho)$ the sequences $\{\Gamma_{f_n}(t)\}$, $\{\beta_{f_n}(t)\}$ (which are defined at least for n large enough) is definitively constant, then define $\Gamma_f(t) = \Gamma_{f_n}(t)$, $\beta_f(t)\beta_{f_n}(t)$ for n large. This may not happen for each t. However, in this case, it is easily seen that there exists $t_0 < \rho$ such that

- $\{\Gamma_{f_n}(t)\}$, $\{\beta_{f_n}(t)\}$ are definitively constant for $t < t_0$,
- $\beta_{f_n}(t)$ is definitively constant for $t \in [t_0, t)$,
- Γ_{f_n}, for n large enough, are constant on $[t_0, \rho_{f_n})$, $h_{\Gamma_{f_n}}(t_0)$ is definitively constant (say equal to i) and the sequence $\{\Gamma_{f_n}(t_0)\}_{n \geq n_0}$ for n_0 large enough is a Cauchy sequence in P_i.

Using the completeness of the P_i's a limit for $\{f_n\}$ is easily constructed.

□

Let us show that $\mathbb{F}$ is geodesic. We will need a notion of restriction of a P-geodesic Γ to a closed subinterval. For each i and any pair of points $q, q' \in P_i$ choose a geodesic $\gamma_{q,q'}$ which connects them. Suppose that Γ has domain $\mathcal{I} = \{I_a\}$, where $\mathcal{I}$ is an almost filling of $[0,l]$. Consider some $0 \leq x \leq l$. First, we define the domain of $\Gamma|_{[0,x]}$ to be $\mathcal{J} = \{J \cap [0,x) : J \in \mathcal{I}$ and $J \cap [0,x) \neq \emptyset\}$. If $J \in \mathcal{J}$ denote by $\hat{J}$ the only interval in $\mathcal{I}$ such that $J = \hat{J} \cap [0,x)$. Define $\Gamma|_{[0,x]}(J) = \gamma_{r_h, \Gamma(\hat{J})}(l(J))$, where $h = h_\Gamma(\hat{J})$.

We can now define, for $f \in \mathbb{F}$, its $\mathbb{F}-$restriction to $[0,x)$ $f\|_{[0,x)}$, for $0 \leq x \leq \rho_f$. We set, for $t \in [0,x)$ and in the domain of Γ_f, $\Gamma_{f\|_{[0,x)}}(t) = \Gamma|_{[0,x]}(t)$ and $\beta_{f\|_{[0,x]}}(t) = \beta_f(t)$ ($\rho_{f\|_{[0,x)}} = x$).

We are finally ready to describe a geodesic between $f, g \in \mathbb{F}$. If $d(f,g)$ is given by the formula in case (b), then γ can be easily checked to be a geodesic parametrized by arc length between f and g, where

$$\gamma(t) = \begin{array}{ll} f\|_{[0,\rho_f - t)} & \text{if } 0 \leq t \leq \rho_f - s \\ g\|_{[0,2s-\rho_f+t)} & \text{if } \rho_f - s \leq t \leq (\rho_f - s) + (\rho_g - s) \end{array}$$

If $d(f,g)$ is given by the formula in case (a), set $\delta = \gamma_{\Gamma_f(s),\Gamma_g(s)}$.

Set $i = h_{\Gamma_f}(s)$, $u = u(f,g)$, $v = v(f,g)$ and $d = d_{P_i}(\Gamma_f(s), \Gamma_g(s))$. The geodesic γ between f and g is given by

$$\gamma(t) = \begin{array}{ll} f\|_{[0,\rho_f - t)} & \text{if } 0 \leq t \leq \rho_f - s - u \\ f\|_{[0,s+t_1)} * f^{\delta(t)} & \text{if } \rho_f - s - u \leq t \leq \rho_f - s - u + d \\ g\|_{[0,2s+t_2+t_1-d-\rho_f+t)} & \text{if } \rho_f - s - u + d \leq t \leq \\ & (\rho_f - s) + (\rho_g - s) + d - u - v \end{array}$$

We will call the geodesics we just described explicit geodesics.

In order to prove that $\mathbb{F}$ is tree-graded, we have to find a candidate set of pieces. For $i \in I$ denote by $w_i \in \mathcal{W}$ the class in $\mathcal{W}$ of a P-geodesic Γ with associated almost-filling (of $[0,1]$) simply $\{[0,1)\}$ with $\Gamma(0) = x$ for some $x \in P_i$, $d(x, r_1) = 1$. If $f, g \in \mathbb{F}$ set $f \leq g$ if their separation moment is ρ_f (it actually is a partial order). Given $f \in \mathbb{F}$, $i \in I$ and $\beta < \alpha(w_i)$, set

$$P(f,i,\beta) = \{g \in \mathbb{F} : f \leq g, \beta_g(\rho_f) = \beta, \text{ and, if } f < g, [\Gamma_g^{\rho_f,\rho_g}] = w_i, [\rho_f, \rho_g) \in \mathcal{I}_g\}.$$

Each $P(f,i,\beta)$ is easily seen to be isometric to P_i (the isometry $P_i \to P(f,i,\beta)$ is given by $x \mapsto f * f^{x,\beta}$). Let $\mathcal{P}$ be the set of all $P(f,i,\beta)$'s. We want to show that $\mathbb{F}$ is tree-graded with respect to $\mathcal{P}$. We will use the characterization of tree-graded spaces given by Theorem 4.18. More precisely, we will use the version stated in Remark 4.20.

First, notice that each $P \in \mathcal{P}$ is geodesic and complete, being isometric to some P_i. In particular, they are closed in $\mathbb{F}$.

Also, it is readily checked that each non-trivial explicit geodesic intersects a piece in a non-trivial subgeodesic. So, each geodesic triangle whose sides are explicit geodesics which intersect each $P \in \mathcal{P}$ in at most one point is trivial. Therefore, if we find a projection system for $\mathcal{P}$, by Lemma 4.21 we are done.

Consider $P = P(f,i,\beta) \in \mathcal{P}$. For each $r \in \mathbb{F}$ define $\pi_P(r)$ to be the first point on the explicit geodesic between r and f. It is obvious that $(P'1)$ holds.

The following claim can be checked directly.

Claim. Suppose that $\pi_P(r_1) \neq \pi_P(r_2)$. Then the explicit geodesic from r_1 to r_2 is obtained concatenating the explicit geodesics from r_1 to $\pi_P(r_1)$, from $\pi_P(r_1)$ to $\pi_P(r_2)$ and from $\pi_P(r_2)$ to r_2.

In particular, $d(r_1,r_2)=d(r_1,\pi_P(r_1))+d(\pi_P(r_1),\pi_P(r_2))+d(\pi_P(r_2),r_2)$, that is, $(P'2)$.

To conclude the proof that $\mathbb{F}$ is tree-graded, we are left to show (T_1). Consider $P(f,i,\beta)$ and $P(g,j,\delta)$. First of all $P(f,i,\beta)$ can have a point in common with $P(g,j,\delta)$ only if $f\leq g$ or vice versa. Let us consider the case $f<g$ (the case $g<f$ is of course analogous, so we will be left to deal only with the case $f=g$). If $h\in P(f,i,\beta)\cap P(g,j,\delta)$ (in particular $h\geq g>f$), then $\mathcal{I}_h$ contains $[\rho_f,\rho_h)$. If we also had $h>g$, $\mathcal{I}_h$ would contain $[\rho_g,\rho_h)$, which is different from $[\rho_f,\rho_h)$, but their intersection is not empty, a contradiction. This readily implies that if $f\in P(f,i,\beta)\cap P(g,j,\delta)$, then $\rho_h=\rho_g$, and so we must have $h=g$.

In the case $f=g$, it is clear that if $i\neq j$ or $\beta\neq\delta$ then $f=g$ is the only point in $P(f,i,\beta)\cap P(g,j,\delta)$.

In order to prove the theorem, we are left to show that for each $w\in\mathcal{W}$, $F^{-1}_{\mathbb{F},p}(w)$ has cardinality $\alpha(w)$, for some choice of pairs identifications. Choose the identification $(P_i,r_i)\to(P(f,i,\beta),f)$ to be $x\mapsto f*f^{x,\beta}$. Recall that we fixed for each i and $x\in P_i$ different from r_i an isometry ϕ_x of P_i which maps x to r_i. These isometries, together with the already fixed identifications, yield a choice of pairs identifications, which is the one we will use.

Notice that each equivalence class in $\mathcal{Y}(\mathbb{F},p)$ has a representative which is an explicit geodesic, by the fact that there is an explicit geodesic connecting each pair of points in $\mathbb{F}$ (clearly, geodesics with the same endpoints have the same initial pattern). Therefore, in what follows we are allowed to restrict to considering explicit geodesics only.

Consider any $f\in\mathbb{F}$. There can be 4 kinds of explicit geodesics starting from f, which are listed below.

1. Explicit geodesics γ such that, for each $\epsilon>0$ in the domain of γ, $f<\gamma(\epsilon)$ and $\beta_{\gamma(\epsilon)}$ is not constant in a neighborhood of ρ_f. In this case $F_{\mathbb{F},f}([\gamma])=[\Gamma^{\rho_f,\rho_f+\epsilon}_{\gamma(\epsilon)}]$ for ϵ as above.

2. Explicit geodesics as in point (1) except that $\beta_{\gamma(\epsilon)}$ is constant in a neighborhood of ρ_f. There is one element in each $F^{-1}_{\mathbb{F},f}(w)$ which can be represented by this kind of explicit geodesics.

3. Explicit geodesics γ such that $F_{\mathbb{F},f}([\gamma])=[-\Gamma_f]$.

4. Other explicit geodesics: in this case there exists an interval in $\mathcal{I}_f$ of the kind $[t,\rho_f)$ such that $F_{\mathbb{F},f}([\gamma])=[-\Gamma_f]$, for any $x\in P_i$, $x\neq r_i$, where $i=h_{\Gamma_f}(t)$.

Let G be the set of equivalence classes in $\mathcal{Y}(\mathbb{F},f)$ of explicit geodesics of type (1). We claim that for each $w\in\mathcal{W}$, the map $H_w:(F^{-1}_{\mathbb{F},f}(w)\cap G)\mapsto\alpha(w)$ given by $[\gamma]\mapsto\beta_{\gamma(\epsilon)}(\rho_f)$ (for any $\epsilon>0$ in the domain of γ) is injective and the image differs from $\alpha(w)$ for at most one element.

If this holds, as geodesics of type $(2)-(4)$ accounts for finitely many elements in each $F^{-1}_{\mathbb{F},f}(w)$ and each $\alpha(w)$ is infinite, we are done.

We are left to prove the claim. Let us prove "almost-surjectivity" first. Suppose $w = [\Gamma]$ for some P-geodesic Γ with domain the almost filling of $[0, l]$ $\mathcal{I}$. For each $\kappa < \alpha(w)$, there exists an element $g(\kappa)$ of $\mathbb{F}$ such that

- $\rho_{g(\kappa)} = l$,
- $\mathcal{I}_{g(\kappa)} = \mathcal{I}$,
- $\Gamma_{g(\kappa)} = \Gamma$,
- $\beta_{g(\kappa)}$ is constantly k.

We have that the explicit geodesic γ from f to $f * g(\kappa)$ is of type (1) for each but at most 1 value of k. As clearly γ is contained in $F^{-1}_{\mathbb{F},f}(w)$ and $H_w(\gamma) = \kappa$, "almost-surjectivity" is proved.

For what regards injectivity, if $H_w(\gamma_1) = H_w(\gamma_2)$, by the fact that the function β_*'s are piecewise constant from the right there exists $\epsilon > 0$ such that $\beta_{\gamma_1(\epsilon)} = \beta_{\gamma_2(\epsilon)}$. It is easily seen that γ_1 and γ_2 have the same pattern until ϵ.

□

Theorem 7.6. *Suppose that $\mathbb{G}$ is tree-graded with trivial transversal trees and that each of its pieces is isometric to one of the P_i's. Also, suppose that for each $p \in \mathbb{G}$ there exists a choice of pairs identifications such that:*

1. *$F^{-1}_{\mathbb{G},p}(w)$ has cardinality less or equal than $\alpha(w)$ for each $w \in \mathcal{W}$. Then $\mathbb{G}$ admits an isometric embedding into $\mathbb{F}_\alpha$.*

2. *$F^{-1}_{\mathbb{G},p}(w)$ has cardinality exactly α_w for each $w \in \mathcal{W}$ and $p \in \mathbb{G}$. Then $\mathbb{G}$ is isometric to $\mathbb{F}_\alpha$.*

Let us sketch the proof of this.

Proof. The proof of point (2) is the same as the proof of Theorem 6.42, with the only difference that the bijection b as in the proof of that theorem exists not because of Proposition 6.38, but because, for each $w \in \mathcal{W}$, $|F^{-1}_{\mathbb{G},p}(w)| = \alpha(w)$ and $\alpha(w)$ is infinite.

The proof of point (1) can be done in the same way, without requiring the "partial" isometric embeddings to preserve fullness.

□

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